Midterm 2

UCLA: Math 115AH, Fall 2018

Instructor: Jens Eberhardt
Date: 19 November 2018

- This exam has 3 questions, for a total of 18 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: [redacted]

ID number: [redacted]

Question	Points	Score
1	4	0
2	6	2
3	8	6
Total:	18	8

med: 8

Good Luck!

1. Let F be a field, V be a vector space over F and $W \subset V$ a ~~subset~~ subspace of V. Let

$\pi(\lambda) =$ $$\pi : V^* \to W^*, \underset{\underset{V^*}{\cap}}{\lambda} \mapsto (W \to F, \underset{\underset{W \in V}{\cap}}{w} \mapsto \lambda(w)).$$

denote the restriction map. This is a linear map. Now consider the map

$$T : \ker(\pi) \to (V/W)^*, \lambda \mapsto (V/W \to F, v + W \mapsto \lambda(v)).$$

(a) (2 points) Show that T is well defined. Namely, show that for all $\lambda \in \ker(\pi)$ and $v, v' \in V$ with $v + W = v' + W$ we have

$$\lambda(v) = \lambda(v').$$

(b) (2 points) Give an explicit formula for the inverse of T.

$\ker(\pi) = \{\lambda \in V^* \mid \lambda(w) \neq 0 \ \forall w \in W\}$

$= W^0$!!?? ✓

$T : W^0 \to V/W^*, \lambda \mapsto (V/W \to F, v+W \mapsto \lambda(v))$.

0/2 a) we know ~~that~~ from quotient space properties that $v+W = v'+W$ iff $v - v' \in W$, so that we can write

if $g \in V/W^*$?

~~$g(v+W) = g\lambda(v)$~~ — wrong! because this is what we are trying to prove!

$\boxed{\lambda(v') = g(v'+W) \equiv g(v+W) = \lambda(v)}$

0/2 b) let $h \in (V/W)^*$, so that $h(v+W) = \lambda(v)$ $\forall v \in V$ what is λ?

$T^{-1} : (V/W)^* \to \ker(\pi)$

$h \mapsto \lambda$

a) $\lambda \in \ker(\pi)$, $v, v' \in V$ s.t. $v+W = v'+W$,

$\lambda(v) \overset{?}{=} \lambda(v')$

$v+W = v'+W \iff v - v' \in W$

$\iff v - v' = w$ for some $w \in W$

$\lambda(v) = \lambda(v'+w) = \lambda(v') + \overset{0}{\cancel{\lambda(w)}} =$

$\lambda \in \ker(\pi) \Rightarrow \pi(\lambda) = 0$ (meaning the zero in W^*)

$\Rightarrow \pi(\lambda)(w) = 0 \ \forall w \in W$

$\Rightarrow \lambda(w) = 0 \ \forall w \in W$

b) $T^{-1}: (V/W)^* \longrightarrow \ker(\pi)$

$TT^{-1}: (V/W)^* \longrightarrow (V/W)^*$

$f \longmapsto f.$

wait!

$f \in (V/W)^*$

$TT^{-1}f = f.$

$(TT^{-1}f)(v+W) = f(v+W) \quad \forall v \in V$

$= T(T^{-1}f)(v+W)$

$= (T^{-1}f)(v) = f(v+W) = f(\pi_0(v)) \quad \pi_0: V \to V/W$

$v \longmapsto v+W$

$T^{-1}f = \underline{f \circ \pi_0} = \underline{\pi_0^* f}.$

both fine

also, try verifying that this is indeed the inverse!

2. Let F be a field and V be a vector space over F. Recall that for an endomorphism $T \in \mathrm{End}(V)$ and $\lambda \in F$ we denote
$$E_\lambda(T) = \ker(T - \lambda \,\mathrm{id}_V) \subseteq V.$$
Now let $S, T \in \mathrm{End}(V)$ be endomorphisms of V such that $TS = ST$. So S and T *commute*.
   (a) (2 points) Show that for $\lambda \in F$ the space $E_\lambda(T)$ is invariant under S.
   (b) (2 points) Let $\lambda \in F$ such that $\dim(E_\lambda(T)) = 1$. Show that there is $\lambda' \in F$ such that
$$E_\lambda(T) \subseteq E_{\lambda'}(S).$$
   (c) (2 points) Now assume that T is diagonalizable and all eigenspaces of T have dimension one. Show that also S is diagonalizable.

2/2 a) let $v \in \ker(T-\lambda id_V)$

then $S(T-\lambda id_V)v = S(0) = 0$

commute + identity pass through every thing $= (T-\lambda id_V)(S(v)) = 0$, or $\boxed{S(v) \in \ker(T-\lambda id_V)} = E_\lambda(T)$

so $S(v) \subset E_\lambda(T) \Rightarrow \boxed{S(E_\lambda(T)) \subset E_\lambda(T)}$ ✓

0/2 b) let $v \in E_\lambda(T)$,

$(T-\lambda id_V)v = 0$ show $(S-\lambda' id_V)v = 0$.

$-S\,T(v) - S(\lambda\, id_V)(v) = 0$.

$TS(v) - \lambda(S(v)) = 0$.

0/2 c) T diag, all $E_{\lambda_i}(T)$ have dim 1, show S diagonalizable.

let $\beta =$

a) Prove $E_\lambda(T)$ is S inv.

Pf: $(\forall v \in E_\lambda(T),\ S(v) \in E_\lambda(T) \Leftrightarrow T(S(v)) = \lambda S(v))$

$(T \circ S)(v) = (S \circ T)(v)$

$= S(T(v))$

$= S(\lambda(v))$

$= \lambda S(v)$

$S(v) \in E_\lambda(T)$ ✓

b) $\dim(E_\lambda(T)) = 1 \Rightarrow E_\lambda(T) \subseteq E_{\lambda'}(S)$ for some $\lambda' \in F$

Pf: Let $v \in E_\lambda(T)$, $v \neq 0$

Then $E_\lambda(T) = \text{span}(\{v\})$. (1-Dim $E_\lambda(T)$)

By (a), $S(v) \in E_\lambda(T) \Rightarrow S(v) = \lambda' v$ for some $\lambda' \in F$

$\Rightarrow v \in E_{\lambda'}(S) \Rightarrow E_\lambda(T) \subseteq E_{\lambda'}(S)$

c) T diag & all $E_{\lambda_i}(T)$ have dim $= 1$, WTS: S diag.

Approaches: 1) $\sum_i \dim(E_{\lambda_i}(T)) = \dim(V)$

2) V has a basis of eigenvectors of S.

Pf: (2)

Since T diag, $\exists$ basis for V $\{v_1, \cdots, v_n\}$ s.t. $v_i \in E_{\lambda_i}(T)$.

$\overset{\text{by b)}}{\Rightarrow} v_i \in E_{\lambda_i'}(S)$ $\{v_1, \cdots, v_n\}$ is still a basis for V and it consists of eigenvectors of S. $\Rightarrow S$ diag.

(1)

$\dim(V) \geq \sum_\mu E_\mu(S) \geq \sum E_{\lambda'}(S) \geq \sum E_\lambda(T) = \dim(V)$

λ' came from λ of T

So $\sum_\mu E_\mu(S) = \dim(V) \Rightarrow S$ diag.

3. Let V be an inner product space over $F = \mathbb{R}$ where the inner product is denoted by $\langle x, y\rangle$ for $x, y \in V$. Consider the map

$$\ell : V \to V^*, x \mapsto (V \to F, y \mapsto \langle x, y\rangle).$$

(a) (2 points) Show that ℓ is linear.

(b) (2 points) Show that ℓ is injective.

(c) (2 points) Assume that V is finite dimensional. Let $\{v_1, \ldots, v_n\} \subset V$ be an orthonormal basis of V. Denote by $\{v_1^*, \ldots, v_n^*\} \subset V^*$ the dual basis. Show that $\ell(v_i) = v_i^*$ for all $1 \le i \le n$.

(d) (2 points) Describe how you could conversely use an isomorphism $T : V \to V^*$ to construct a function $V \times V \to F$ and under which condition this would be a inner product. (You do not need to prove anything here).

a) $\ell(cx+y)(z)$

$= \langle cx+y, z\rangle$

$= c\langle x, z\rangle + \langle y, z\rangle$

$= c\ell(x)(z) + \ell(y)(z) \Rightarrow \ell(cx+y) = c\ell(x) + \ell(y) \Rightarrow$ linear

✓ /2

b) injective: $\ell(x) = 0$ iff $x = 0$ (for linear maps)

$\Leftrightarrow \ell(x)(z) = 0 \quad \forall z$

$\Leftrightarrow \langle x, z\rangle = 0, \forall z \Leftrightarrow \boxed{x = 0}$ injective

/2 ✓

c) $\ell(v_i)(v_j)$

$= \langle v_i, v_j\rangle$

$= \delta_{ij}$

$= v_i^*(v_j) \Rightarrow \ell(v_i) = v_i^*$, bc all are linear maps & if linear maps agree on these of basis, they are identical (Thm 2.6)

/2 Nice ✓

d) 0

$T: V \to V^*$

construct: $b: V \times V \to \mathbb{R}$ s.t. it's inner product.

$(x, y) \mapsto T(x)(y)$

inner product if follows inner prod axioms.

- $\forall a \in F, x_1, x_2 \in V$

  $b(ax_1 + x_2, y) = a\,b(x_1, y) + b(x_2, y)$ ✓

- $b(0,0) = 0$

  $b(x,x) > 0 \quad \forall x \in V \neq 0 \Leftrightarrow [T(x)](x) \geq 0$

- $\forall x, y \in V \quad b(x,y) = b(y,x)$ ☆ bc $F = \mathbb{R}$, not $\mathbb{R}$ or $\mathbb{C}$

  $\Rightarrow (T(x))(y) = (T(y))(x)$

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