Midterm 1

UCLA: Math 115AH, [redacted] 2018

FALL

Instructor: Jens Eberhardt
Date: 22 October 2018

- This exam has 4 questions, for a total of 24 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	6	6
2	6	6
3	6	6
4	6	6
Total:	24	24

Good Luck!

Jens

1. Let U, V, W be finite-dimensional vector spaces over a field F, and let $T : U \to V$ and $S : V \to W$ be linear maps. Assume that T is injective, S is surjective and $ST = 0$.

(a) (3 points) Show that

$$\dim V \geq \dim U + \dim W$$

(b) (3 points) Show that

$$\dim V = \dim U + \dim W \iff \operatorname{im}(T) = \ker(S).$$

Hint: Use the rank-nullity formula. Also, what does $ST = 0$ imply for $\operatorname{im}(T)$ and $\ker(S)$?

a. Given $ST = 0 \Rightarrow \operatorname{im}(T) \subset \ker(S)$ [illegible] $\forall u \in U$,

$T(u) \in \ker(S)$, [illegible] $S($[illegible]$) = 0$

Using rank-nullity we get

$\dim(U) = \operatorname{rank}(T) + \operatorname{nullity}(T)$

$\dim(V) = \operatorname{rank}(S) + \operatorname{nullity}(S)$

as S surjective $\Rightarrow \operatorname{Im}(S) = W$, $\operatorname{rank}(S) = \dim(W)$

and from before, as $\operatorname{im}(T) \subset \ker(S) \Rightarrow \operatorname{rank}(T) \leq \operatorname{nullity}(S)$

$\Rightarrow \dim(V) \geq \dim(W) + \operatorname{rank}(T)$

as T injective $\Rightarrow \forall u \in U, \exists! v \in V$ s.t. $T(u) = v$

$\Rightarrow \ker(T) = \{0\}$, $\dim(U) = \operatorname{rank}(T)$

$\Rightarrow \dim(V) \geq \dim(W) + \dim(U)$

b. "$\Leftarrow$" Assume $\operatorname{Im}(T) = \ker(S)$, then $\operatorname{rank}(T) = \operatorname{nullity}(S)$

$\Rightarrow \dim(V) = \dim(W) + \operatorname{rank}(T)$

$\Rightarrow \dim(V) = \dim(W) + \dim(U)$

"$\Rightarrow$" Assume $\dim(V) = \dim(W) + \dim(U)$, then

$\dim(V) = \dim(W) + \operatorname{rank}(T)$ and given $\dim(U) = \operatorname{rank}(T)$

$\dim(V) = \operatorname{rank}(S) + \operatorname{nullity}(S) \Rightarrow \operatorname{rank}(T) = \operatorname{nullity}(S)$

$\Rightarrow$ as $ST = 0$ [illegible] $\operatorname{im}(T) = \ker(S)$

2. Let F be a field, n a positive integer, $V = F^n$ and

$$W = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mid a_1, \ldots, a_n \in F, \sum_{i=1}^{n} a_i = 0 \right\} \subset V.$$

(a) (4 points) Show that W is a subspace of V.

(b) (2 points) Compute the dimension of V/W and W.

a. $\sum_{i=1}^{n} 0 = 0 \Rightarrow 0_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in W$, thus W contains 0 element

closed: Let $v, u \in W$, $\lambda \in F$, and $v = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $u = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, then

$$\sum_{i=1}^{n} \lambda a_i + b_i = \lambda \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \lambda(0) + 0 = 0$$

$$\Rightarrow \begin{pmatrix} \lambda a_1 + b_1 \\ \vdots \\ \lambda a_n + b_n \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \lambda v + u \in W$$

Thus W closed under $+$, $\cdot$, and $0_V \in W$

$\Rightarrow W$ a subspace of V.

b. $\beta = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ \vdots \\ 0 \\ -1 \end{pmatrix} \right\}$, $|\beta| = n-1$

$\dim(V/W) = 1$, $\dim(W) = n-1$

(Would be nice to explain how what you wrote supports your conclusion.)

Find map s.t. $\ker(T) = W$

$T: V \to F$, $x = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mapsto T(x) = \sum a_i$

$\ker(T) = W$, $\underset{\substack{\| \\ n}}{\dim(V)} = \underset{\substack{\| \\ n-1}}{\text{nullity}(T)} + \underset{\substack{\| \\ 1}}{\text{rank}(T)}$

Every vector space can be written as the kernel of some linear transformation; in this one, it is fairly straightforward.

3. (6 points) Let F be a field and V be a finite-dimensional vector space over F. Let β be a basis of V and $0 \neq v \in V$ some non-zero vector. Show that there is a $x \in \beta$ such that

$$(\beta - \{x\}) \cup \{v\}$$

is a basis of V.

You may **not** use the Replacement Theorem.

WTS: $(\beta - \{x\}) \cup \{v\}$ is LI, spans V

$\beta = \{u_1, \ldots, u_n\}$, and $a_1, \ldots, a_n \in F$, then

$$v = \sum_{i=1}^{n} a_i u_i$$

As $0 \neq v \in V = \text{span}(\beta)$, we can say that one of the a_i must be non-zero, and thus w.l.o.g. we say $a_1 \neq 0$, then

$$u_1 = a_1^{-1}\left(\sum_{i=2}^{n} a_i u_i - v\right)$$

Thus I claim $(\beta - \{u_1\}) \cup \{v\}$ is a basis of V.

Pf:

LI: Let $b_1, \ldots, b_n \in F$, then

$$b_1 v + b_2 u_2 + \ldots + b_n u_n = 0$$

$$\Rightarrow b_1\left(\sum_{i=1}^{n} a_i u_i\right) + b_2 u_2 + \ldots + b_n u_n = 0$$

$$\Rightarrow b_1 a_1 u_1 + \sum_{i=2}^{n} (b_1 a_i + b_i) u_i = 0$$

as β is LI $\Rightarrow b_1 a_1 = b_1 a_2 + b_2 = \ldots = b_1 a_n + b_n = 0$

as before, $a_1 \neq 0$, thus $b_1 = 0 \Rightarrow b_2 = b_3 = \ldots = b_n = 0$

Therefore $b_1 v + b_2 u_2 + \ldots + b_n u_n = 0$ only for $b_1 = b_2 = \ldots = b_n = 0$

$\Rightarrow (\beta - \{u_1\}) \cup \{v\}$ is LI.

Nice!

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Span: Let $w \in V$ arbitrary, then for some $\lambda_i \in F$

$$\lambda_1 u_1 + \ldots + \lambda_n u_n = w$$

$$\Rightarrow \lambda_1 a_1^{-1}\left(\sum_{i=1}^{n} a_i u_i - v\right) + \lambda_2 u_2 + \ldots + \lambda_n u_n = w$$

$$\Rightarrow \lambda_1 a_1^{-1} v + \lambda_1 a_1^{-1} \sum_{i=2}^{n} (\lambda_i + a_i) u_i = w \Rightarrow (\beta - \{u_1\}) \cup \{v\}$$ generates V

4. Let F be a field. Recall that for a set S the set of functions from S to F

$$\mathcal{F}un(S, F) = \{\alpha : S \to F\}$$

forms a vector space over F with addition and scalar multiplication given by

$$\alpha_1 + \alpha_2 : S \to F,\ x \mapsto \alpha_1(x) + \alpha_2(x) \text{ and}$$
$$t\alpha : S \to F,\ x \mapsto t\alpha(x)$$

for $\alpha, \alpha_1, \alpha_2 \in \mathcal{F}un(S, F)$ and $t \in F$.

Now let X, Y be sets and $f : X \to Y$ a function.

(a) (4 points) Show that the map

$$f^* : \mathcal{F}un(Y, F) \to \mathcal{F}un(X, F), \alpha \mapsto \Big[\alpha f : X \to F,\ x \mapsto \alpha(f(x))\Big]$$

is linear. For $\alpha \in \mathcal{F}un(Y, F)$, $f^*(\alpha)$ is called the *pullback* of α along f.

(b) (2 points) Assume that f is invertible. Show that f^* is invertible by writing down its inverse. Do not forget to show that your inverse is an inverse!

a. Linear: Let $h, g \in \mathcal{F}un(Y, F)$, $\lambda \in F$, then

$f^*(\lambda h + g) = (\lambda h + g)f = \lambda hf + gf = \lambda f^*(h) + f^*(g)$

Therefore f^* is linear.

b. $f^{*-1} : \mathcal{F}un(X, F) \longrightarrow \mathcal{F}un(Y, F),\ \beta \mapsto [\beta f^{-1} : Y \to F,\ y \mapsto \beta(f^{-1}(y))]$

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Let $g \in \mathcal{F}(Y, F)$

$f^{*-1}(f^*(g)) = f^{*-1}(gf) = f^{*-1}(fg) = f^{-1}(fg) = (f^{-1}f)g$ /2

$= (Id \cdot g) = g$, where Id is identity map

$f^*(f^{*-1}(g)) = f^*(f^{-1}g) = f(f^{-1}g) = (ff^{-1})g = g$ Nice

Thus f^{*-1} is inv of f^*.

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