Midterm 1

UCLA: Math 115AH, ~~Spring~~ 2018
FALL

Instructor: Jens Eberhardt
Date: 22 October 2018

- This exam has 4 questions, for a total of 24 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: [redacted]

ID number: [redacted]

Question	Points	Score
1	6	1
2	6	6
3	6	6
4	6	6
Total:	24	19

med: 16

Good Luck!

Jens

1. Let U, V, W be finite-dimensional vector spaces over a field F, and let $T : U \to V$ and $S : V \to W$ be linear maps. Assume that T is injective, S is surjective and $ST = 0$.

$U \xrightarrow{T} V \xrightarrow{S} W$ T inj S surj $ST=0$. Prove

(a) (3 points) Show that

$$\dim V \overset{\geq}{\cancel{\leq}} \dim U + \dim W.$$

(b) (3 points) Show that

$$\dim V = \dim U + \dim W \iff \operatorname{im}(T) = \ker(S).$$

Hint: Use the rank-nullity formula. Also, what does $ST = 0$ imply for $\operatorname{im}(T)$ and $\ker(S)$?

a) for T, we have

$\dim(U) = \operatorname{rank}(T) + \operatorname{null}(T)$, but $\operatorname{Rank}(T) \leq \dim(U)$
for S, we have
$\dim(V) = \operatorname{rank}(S) + \operatorname{null}(S)$

$\operatorname{rank}(S) \leq \dim(V)$
$\operatorname{null}(T) \leq \dim(V)$
$\operatorname{null}(S) \leq \dim(W)$

Lemma:
$S(T(x)) = 0 \quad \forall x$
$T(x) \in N(S)$
$\operatorname{im}(T) \subseteq N(S)$
$\boxed{\operatorname{rank} T \leq \operatorname{null}(S)}$

from lemma:

$\dim(U) = \operatorname{rank}(T) + \operatorname{null}(T)$
$\leq \operatorname{null}(S) + \operatorname{null}(T) \leq \dim(V) + \dim(W)$

$\Rightarrow \dim(V) \geq \dim(U) - \dim(W)$
$\geq \dim(U) + \dim(W)$?? since dimensions ≥ 0.

Hint: Use all the informati

a) T inj: $\ker(T) = 0 \iff \boxed{\operatorname{rank}(T) = \dim(U)} \neq \dim(V)$
Surj: $\boxed{\operatorname{rank}(S) = \dim(W)}$

$\operatorname{rank}(T) \leq \operatorname{null}(S)$
$(\dim(U) \leq \dim(V) - \dim(W))$

b) if $\operatorname{im}(T) = \ker(S)$
$\operatorname{rank}(T) = \operatorname{null}(S)$.

Clear

"$\Rightarrow$": We already know $\operatorname{im}(T) \subseteq \ker(S)$, now we ~~also~~ found that they have = dimension, & we know that a vector space can have only ~~one~~ subspace w/ equal dimension: itself ~~…~~ $\Rightarrow$ ~~one~~ is $\operatorname{im}(T) = \ker(S)$.

2. Let F be a field, n a positive integer, $V = F^n$ and

$$W = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mid a_1, \ldots, a_n \in F, \sum_{i=1}^{n} a_i = 0 \right\} \subset V.$$

(a) (4 points) Show that W is a subspace of V.

(b) (2 points) Compute the dimension of V/W and W.

To show subspace
1: $0_V \in W$
2: closure
3: $W \subseteq V$.

a) for $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in W$, since $\sum_{i=1}^{n} 0 = 0$

closure let $x, y \in W$, $c \in F$

$cx + y = c\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} cx_1 + y_1 \\ \vdots \\ cx_n + y_n \end{pmatrix}$, but $\sum_{i=1}^{n} cx_i + y_i = c\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i = 0$

since $x, y \in W$ $\boxed{cx + y \in W}$

Also $W \subseteq V$, so W is a subspace of V. (Complete your argument)

b) $\dim(V) = \dim(V/W) + \dim(W) \to \dim(V) =$ known, so that we only need to find $\dim(W)$:

claim: $\beta = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis for W w/ dim $n-1$.

Pf: let $w \in W$, then we can write $w = (w_1, w_2, \ldots, w_n)$, as

$a_1 = w_1$

$a_2 = w_2 + a_1$

$a_3 = w_3 + a_2$

$\vdots$

$a_{n-1} = w_{n-1} + a_{n-2}$

Compute $\dim(V/W)$, $\dim(W)$
$\dim(V) = \dim(V/W) + \dim(W)$
OR
find LT w/ nullspace $= W$
$T: V \to F$, $x = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \longmapsto \sum a_i$.
$\ker(T) = W$. $\dim(V) = \underbrace{\dim(W)}_{n-1} + \operatorname{rank}(T)$

so that β is a basis for W. w/ $\dim(W) = n-1$

Also lin. ind.

$\dim(W) = n-1$

$d(V/W) = n - \dim(W) = 1$

every ~~subspace~~ vectorspace can be written as the kernel of some linear transformation.

25 left

3. (6 points) Let F be a field and V be a finite-dimensional vector space over F. Let β be a basis of V and $0 \neq v \in V$ some non-zero vector. Show that there is a $x \in \beta$ such that

$$\beta - \{x\} \cup \{v\} \equiv \gamma$$

is a basis of V.

You may **not** use the Replacement Theorem.

prove $\exists\ x \in \beta$ s.t $\beta - \{x\} \cup \{v\}$ is basis

Linear ind.

let $v \in V$ so that for $\beta = \{x_1, \dots, x_n\}$ ✓

$v = \sum_{i=1}^{n} a_i x_i$, $a_i \in F$, $n = \dim(V)$

since $v \neq 0$, at least one $a_i \neq 0$, say a_1, ✓

then we can write $v = a_1 x_1 + \sum_{i=2}^{n} a_i x_i$, $a_1 \neq 0$.

If we then write, for $b \in F$

$0 = bv + \sum_{i=2}^{n} a_i x_i$

$= a_1 b_1 x_1 + 2 \sum_{i=2}^{n} a_i x_i$,

$a_1 b_1$ and $2a_i$ must all $= 0$,

but $a_1 \neq 0$, so $b = 0$,

so that $\{v\} \cup \beta - \{x_1\}$ is LI

Then from book: LI set of vectors $\subseteq V$ ~~with~~ w/ n vectors, where $n = \dim(V)$ ~~thus~~ generates V.

$\beta - \{x\} \cup \{v\}$ is therefore a basis.

proving by generation & same ~~nums~~ num of vectors instead.

Pf: $|\gamma| = |\beta| + 1 - 1 = |\beta|$.

prove γ generates V.

sufficient to prove $\beta \subset \mathrm{span}(\gamma)$. clear

For $u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n\}$ done

For u_k:

$u_k = a_k^{-1}(v - \sum_{i \neq k} a_i u_i)$

so that $\beta \subset \mathrm{span}(\gamma)$.

/6

4. Let F be a field. Recall that for a set S the set of functions from S to F

$$\mathcal{F}un(S,F) = \{\alpha : S \to F\}$$

forms a vector space over F with addition and scalar multiplication given by

$$\alpha_1 + \alpha_2 : S \to F,\ x \mapsto \alpha_1(x) + \alpha_2(x) \text{ and}$$
$$t\alpha : S \to F,\ x \mapsto t\alpha(x)$$

for $\alpha, \alpha_1, \alpha_2 \in \mathcal{F}un(S,F)$ and $t \in F$.

Now let X, Y be sets and $f : X \to Y$ a function.

(a) (4 points) Show that the map

$$f^* : \mathcal{F}un(Y,F) \to \mathcal{F}un(X,F), \alpha \mapsto \left[\alpha f : X \to F,\ x \mapsto \alpha(f(x))\right]$$

is linear. For $\alpha \in \mathcal{F}un(Y,F)$, $f^*(\alpha)$ is called the *pullback* of α along f.

(b) (2 points) Assume that f is invertible. Show that f^* is invertible by writing down its inverse. Do not forget to show that your inverse is an inverse!

$f^*(\alpha_1) + f^*(\alpha_1)(x)$

ex.

$f(x) = x^2$.

$f : x \mapsto x^2$

$f(1) = 1$

a) for $\alpha_1, \alpha_2 \in \mathcal{F}un(Y,F)$ $t \in F$.

$f^*(t\alpha_1 + \alpha_2) = (t\alpha_1 + \alpha_2) f$

$= (t\alpha_1)(f) + \alpha_2(f)$

$= t(\alpha_1(f)) + (\alpha_2 f)$

$= t f^*(\alpha_1) + f^*(\alpha_2)$

composition of functions distribute & associative.

linear

b) f inv $\Leftrightarrow$ bijective $\Leftrightarrow$ $f^{-1}(f(x)) = x$.

$f^{*-1} : \mathcal{F}(X,F) \to \mathcal{F}(Y,F)$

$\beta \in \mathcal{F}(X,F) \to \beta f^{-1}$.

$f^{*-1}(f^*(\alpha)) = f^{*-1}(\alpha f)$

$= \alpha f f^{-1} = \alpha$.

$f^*(f^{*-1}(\beta)) = f^*(\beta f^{-1})$

$= \beta f^{-1} f = \beta$.

$\beta \in \mathcal{F}(X,F)$

6/6