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Problem 1: Let F_3 be the field $F_3 = \{0, 1, 2\}$ with multiplication and addition given by $a + b = (a + b)\text{mod}3$ and $ab = (ab)\text{mod}3$. Let V be the vector space over F_3 consisting of all ordered pairs (a, b) where a and b are both elements of F_3.

Let S be the subset of V given by $S = \{(0, 0), (2, 1), (1, 2)\}$. Show that S is a subspace of V.

S has the $\vec{0}$; $(0,0) \in S$

S is closed under vector addition:

$(0,0) + (2,1) = (2,1) \in S$

$(0,0) + (1,2) = (1,2) \in S$

$(2,1) + (1,2) = (0,0) \in S$

$(1,2) + (1,2)$?

$(2,1) + (2,1) = (1,2) \in S$

$(1,2) + (1,2) = (2,1) \in S$

S is closed under scalar multiplication:

$0 \cdot (a,b) = \vec{0} \in S$ where (a,b) is any vector in S

$1 \cdot (a,b) = (a,b) \in S$

$2 \cdot (2,1) = (1,2) \in S$

$2 \cdot (1,2) = (2,1) \in S$

$2 \cdot (0,0) = (0,0) \in S$

$\therefore$ S is a subspace of V

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Problem 2: Let F be a field, let V and W be vector spaces over F, and let $T : V \to W$ be a linear transformation. Suppose that the zero vector in V, $\vec{0}$, is the only element of $\ker(T)$. Show that T is injective (one-to-one).

(Hint: Let u, v be vectors in V with $T(u) = T(v)$, and then show that $u = v$.)

Let $v \in V$ st. $T(u) = T(v)$

$v = u + w$ where w is another vector in V (if v is different from u)

$T(u) = T(u+w)$

$T(u) = T(u) + T(w)$

So $T(w)$ must be 0 Nice

But the only element in $\ker(T)$ is $\vec{0}$,

so

$v = u + \vec{0}$

$v = u$

$\therefore$ T is injective because no two unique vectors are transformed to the same element in W

Problem 3: (a) Suppose that F is a field in which $1+1=0$. Show that every element of F is its own additive inverse.

Let $a \in F$, Because F has multiplicative identity

$$a+a = a\cdot 1 + a \cdot 1$$
$$= a(1+1)$$
$$= a(0) = \vec{0}$$

(b) There is a field F with four elements: $F = \{0, 1, a, b\}$. In this field, $1+1=0$. Fill out these tables showing what addition and multiplication do, in F. (So since $0+0=0$, there should be a 0 in the upper left corner of the first table, for example. This is a tricky puzzle, but there is enough information to fill the tables out completely.)

Multiplicative identity/inv. is unique

Additive identity/inv. is unique

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

·	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

F

$a+$: $0 \neq a$ 1 ~~X~~

$a = 0$

$a+1 \neq 1 \Rightarrow a = 0$

$a+1 = b$, $b+1 = a$

$a+b$ $a+b = 1$

$a+b$

$a \cdot 1 + b \cdot 1$

$(a+b)1$

$a+b \neq a \neq 0$

$a+b=1$

$a+b=1$

$a \cdot 0 = 0$ 1, b

$a \cdot 1 = a$

$a \cdot a = $

$a \cdot b = $

$a \cdot b \neq b$

$a \neq 1$ $a \cdot b = 1$

$a \cdot a = b$

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Problem 4: (a) Let X be the set of even integers, and let Y be the set of integers which are multiples of 3. Find a bijection $f : X \to Y$.

Let $n \in X$

$$f = \begin{cases} \frac{3n}{2} & \text{if } n > 0 \\ \frac{3n}{2} + 3 & \text{if } n < 0 \end{cases}$$

$$\begin{array}{ccccccccccc} \cdots & -8 & -6 & -4 & -2 & & 2 & 4 & 6 & 8 & \cdots \\ \cdots & -9 & -6 & -3 & 0 & & 3 & 6 & 9 & 12 & \cdots \end{array}$$

What about $n = 0$?

Note: 0 is even
$f = \frac{3n}{2}$

(b) Let X be the set of all positive integers, $X = \{n \in \mathbb{Z} | n > 0\}$, and let Y be the set of all ordered triples of positive integers, $Y = \{(a, b, c,) | a, b, c \in X\}$. So Y includes the points $(2, 7, 10)$, $(8, 3, 8)$, and $(11, 11, 11)$, for example.

Describe a bijection from X to Y. You don't need to give an explicit formula, and drawing pictures is encouraged.

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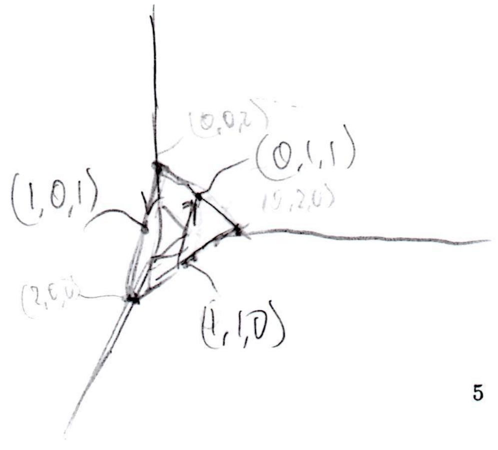
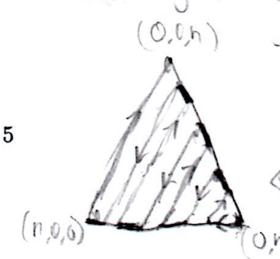
Y forms a section of space as shown below.



Every ordered triple can be iterated through (and assigned a positive integer) by taking successive "triangles" with vertices at $(n,0,0)$, $(0,n,0)$, and $(0,0,n)$. The integers in each triangle can be iterated through by zig-zagging in the pattern shown here.



This pattern will eventually hit all ordered triples of positive integers, so every positive integer will eventually be paired with a triple, which makes this a bijection

Nice