Midterm Solutions

1. (25 pts)

a.) We are given a cylindrical plasma initially with uniform plasma density n_0 with radius r_0 and length L_0. If the electrons are at temperature T_0, the kinetic energy is

$$U = \frac{3}{2}NKT_0 = \boxed{\frac{3}{2}KT_0 n_0 \pi {r_0}^2 L_0} \qquad (5\ pts)$$

b.) Suppose all the electrons move out to a radius r_0 and ions are left inside the cylindrical volume. Outside the cylinder, the electric field is zero. At a radius r inside the cylinder, we can write Gauss' law assuming $L_0 \gg r_0$ and ignoring fringe fields

$$\oiint \vec{E} \cdot d\vec{S} = \frac{Q(r)}{\epsilon_0}$$

$$E_r 2\pi r L_0 = \frac{e n_0 \pi r^2 L_0}{\epsilon_0}$$

$$\boxed{E_r = \frac{e n_0 r}{2\epsilon_0}} \qquad (5\ pts)$$

The total electrostatic potential energy can by integrating the energy density over the volume of the cylinder,

$$U = \oiiint \frac{1}{2}\epsilon_0 E^2 dV = \frac{1}{2}\epsilon_0 \left(\frac{e n_0}{2\epsilon_0}\right)^2 \int_0^{r_0} r^2\, 2\pi r dr L_0$$

$$\boxed{U = \frac{\pi e^2 {n_0}^2 L_0 {r_0}^4}{16\epsilon_0}} \qquad (5\ pts)$$

c.) Equating the electrostatic potential energy with the kinetic energy

$$\frac{\pi e^2 {n_0}^2 L_0 {r_0}^4}{16\epsilon_0} = \frac{3}{2}KT_0 n_0 \pi {r_0}^2 L_0$$

$$r_0{}^2 = 24\frac{\epsilon_0 K T_0}{e^2 n_0} = 24\lambda_D{}^2$$

$$\boxed{r_0 = 2\sqrt{6}\,\lambda_D} \qquad (10\ pts)$$

2. (25 pts)

A particle with charge q and mass m moves in electric and magnetic fields given by

$$\vec{E} = (E_0 + xE')\hat{x}, \quad \vec{B} = B_0\hat{z}$$

where E_0, E, and B_0 are constants.

a.) We can solve the equations of motion exactly with the Lorentz force.

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Component-wise, we find

$$\dot{v}_x = \frac{q}{m}\left(E_0 + xE' + v_y B_0\right)$$

$$\dot{v}_y = -\frac{q v_x B_0}{m}$$

Taking the time-derivative of the x-equation,

$$\ddot{v}_x = \frac{qE'}{m} v_x + \frac{q\dot{v}_y B_0}{m}$$

$$\ddot{v}_x = \frac{qE'}{m} v_x - \left(\frac{qB_0}{m}\right)^2 v_x$$

$$\ddot{v}_x = -\omega^2 v_x, \qquad \omega = \sqrt{\left(\frac{qB_0}{m}\right)^2 - \frac{qE'}{m}}$$

If ω^2 is positive, then this is simple harmonic motion. Therefore, we see gyration if

$$\omega^2 > 0 \;\; \rightarrow \;\; \boxed{\left(\frac{qB_0}{m}\right)^2 > \frac{qE'}{m}} \qquad (8\,pts)$$

A solution for v_x is

$$v_x = v_{\perp 0}\cos(\omega t)$$
$$x = x_0 + \frac{v_{\perp 0}}{\omega}\sin(\omega t)$$

b.) Plugging back into our force equation,

$$\dot{v}_x = \frac{q}{m}\left(E_0 + xE' + v_y B_0\right)$$
$$v_y = \frac{m\dot{v}_x}{qB_0} - \frac{E_0}{B_0} - \frac{xE'}{B_0}$$
$$v_y = -\frac{m}{qB_0}\omega v_{\perp 0}\sin(\omega t) - \frac{E_0}{B_0} - \frac{E'}{B_0}\left(x_0 + \frac{v_{\perp 0}}{\omega}\sin(\omega t)\right)$$
$$v_y = -\frac{1}{\omega_c \omega} v_{\perp 0}\sin(\omega t)\left(\omega^2 + \frac{qE'}{m}\right) - \frac{E_0}{B_0} - \frac{E' x_0}{B_0}$$

$$\omega^2 + \frac{qE'}{m} = \left(\frac{qB_0}{m}\right)^2 - \frac{qE'}{m} + \frac{qE'}{m} = {\omega_c}^2$$

$$\boxed{v_y = -\frac{\omega_c}{\omega} v_{\perp 0}\sin(\omega t) - \frac{E_0 + E' x_0}{B_0}}$$

Particle drifts in $\hat{y}$, which is perpendicular to both $\vec{E}$ and $\vec{B}$ with velocity,

$$\boxed{\vec{v}_D = -\frac{E_0 + E' x_0}{B_0}\hat{y}} \quad (7\,pts)$$

c.) The gyro frequency is

$$\omega = \sqrt{\left(\frac{qB_0}{m}\right)^2 - \frac{qE'}{m}} \quad (5\,pts)$$

d.) If $\frac{qE'}{m} > \left(\frac{qB_0}{m}\right)^2$, the equation of motion for v_x becomes

$$\ddot{v}_x = \lambda^2 v_x, \qquad \lambda = \sqrt{\frac{qE'}{m} - \left(\frac{qB_0}{m}\right)^2}$$

The particle motion is exponential in this case.

$$\boxed{v_x = Ae^{\pm\lambda t}} \quad (5\ pts)$$

3. (25 pts)

A charged particle moves in a magnetic mirror with $v_{\perp 0}$ at $z = 0$. The magnetic field varies with time

$$B_z = B_0(1 + \alpha(t)^2 z^2)$$

a.) The adiabatic invariants are the magnetic moment μ and longitudinal invariant J. (5 pts)

b.) The equation of motion in z is

$$m\ddot{z} = -\mu \frac{\partial B_z}{\partial z}$$

$$\boxed{\ddot{z} = -\frac{2\mu B_0 \alpha(t)^2}{m} z} \quad (5\ pts)$$

c.) We solve our equation of motion for constant α. Taking another time derivative,

$$\ddot{v}_z = -\frac{2\mu B_0 \alpha^2}{m} v_z$$

$$\ddot{v}_z = -\omega^2 v_z, \quad \omega = \sqrt{\frac{2\mu B_0 \alpha^2}{m}}$$

This is simple harmonic motion. We know $v_z(t) = v_{z0}$ and $z(t) = 0$ the time $t = 0$. The solutions are

$$\boxed{v_z(t) = v_{z0}\cos(\omega t)\,, \;\; z(t) = \frac{v_{z0}}{\omega}\sin(\omega t)} \quad (7\ pts)$$

d.) Now we can solve for the second adiabatic invariant.

$$J = \oint v_z dz = \oint v_z \frac{dz}{dt} dt = \oint {v_z}^2 dt$$

$$J = \int_0^{2\pi/\omega} {v_{z0}}^2 \cos^2(\omega t)\, dt = \frac{{v_{z0}}^2}{\omega} \int_0^{2\pi} \cos^2(\omega t)\, d(\omega t) = \pi \frac{{v_{z0}}^2}{\omega} = const$$

Recall the magnetic moment μ is invariant from (a.) and B_0 and m are constants. v_{z0} is the velocity at the origin $z = 0$.

$$\frac{{v_{z0}}^2}{\omega} = \frac{{v_{z0}}^2}{\sqrt{\frac{2\mu B_0 \alpha^2}{m}}} \sim \frac{{v_{z0}}^2}{\alpha} = const$$

$$\frac{v_{z0}(t)^2}{\alpha(t)} = \frac{v_{z0}(0)^2}{\alpha(0)}$$

$$\boxed{v_{z0}(t) = \sqrt{\frac{\alpha(t)}{\alpha(0)}}\, v_{z0}(0)} \quad (4\ pts)$$

After each bounce the parallel velocity changes depending on $\alpha(t)$. The amplitude of oscillation z_{max} can be found by looking at the answers from (c)

$$z_{max} = \frac{v_{z0}}{\omega}$$

Substituting into the boxed equation above,

$$z_{max}(t)\omega(t) = \sqrt{\frac{\alpha(t)}{\alpha(0)}}\, z_{max}(0)\omega(0)$$

$$\boxed{z_{max}(t) = \sqrt{\frac{\alpha(0)}{\alpha(t)}}\, z_{max}(0)} \quad (4\ pts)$$

4. (25 pts)

We are given a distribution function for electrons in phase-space for a 1-D plasma. We drop the subscript x in the following solution.

$$f(v) = n_0 \left(\frac{m}{2\pi K T_0}\right)^{1/2} \exp\left(-\frac{m(v+v_0)^2}{2KT_0}\right) + n_1 \left(\frac{m}{2\pi K T_0}\right)^{1/2} \exp\left(-\frac{m(v-v_1)^2}{2KT_0}\right)$$

Our distribution function can also be written as

$$f(v) = \frac{n_0}{\sqrt{\pi}}\sqrt{\alpha}e^{-\alpha(v-v_1)^2} + \frac{n_1}{\sqrt{\pi}}\sqrt{\alpha}e^{-\alpha(v-v_1)^2}, \qquad \alpha = \frac{m}{2KT_0}$$

a.) In order to compute the density $n(x,t)$ we need to integrate our distribution function over 1D velocity space

$$n(x,t) = \int_{-\infty}^{\infty} f(v)dv$$

It is provided that the integral of a Gaussian function is

$$\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$$

Our integrals take the form

$$\frac{n_1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \sqrt{\alpha}e^{-\alpha(v-v_1)^2}\, dv, \qquad \alpha = \frac{m}{2KT_0}$$

Using u-sub, we can reduce these integrals

$$u = \sqrt{\alpha}(v - v_1), \quad du = \sqrt{\alpha}du$$

$$\frac{n_1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \sqrt{\alpha}e^{-\alpha(v-v_1)^2}\, dv = \frac{n_1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\, du = n_1$$

After integrating each distribution over velocity space, we are left with the number density of that respective distribution function.

$$n(x,t) = \int_{-\infty}^{\infty} f(v)dv = \frac{n_0}{\sqrt{\pi}}\int_{-\infty}^{\infty} \sqrt{\alpha}e^{-\alpha(v+v_0)^2}\,dv + \frac{n_1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \sqrt{\alpha}e^{-\alpha(v-v_1)^2}\,dv = n_0 + n_1$$

$$\boxed{n(x,t) = n_0 + n_1} \qquad (5\,pts)$$

b.) The average velocity is given by,

$$\langle v\rangle = \frac{\int_{-\infty}^{\infty} vf(v)dv}{\int_{-\infty}^{\infty} f(v)dv} = \frac{1}{n(x,t)}\int_{-\infty}^{\infty} vf(v)dv = \frac{1}{n_0 + n_1}\int_{-\infty}^{\infty} vf(v)dv$$

Our integrals now take the form

$$\frac{n_1}{\sqrt{\pi}}\int_{-\infty}^{\infty} v\sqrt{\alpha}e^{-\alpha(v-v_1)^2}\,dv, \qquad \alpha = \frac{m}{2KT_0}$$

We can simplify this integral,

$$\frac{n_1}{\sqrt{\pi}}\int_{-\infty}^{\infty} v\sqrt{\alpha}e^{-\alpha(v-v_1)^2}\,dv = \frac{n_1}{\sqrt{\pi}}\int_{-\infty}^{\infty} (v-v_1)\sqrt{\alpha}e^{-\alpha(v-v_1)^2}\,dv + v_1\frac{n_1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \sqrt{\alpha}e^{-\alpha(v-v_1)^2}\,dv$$

The first integral is 0

$$\frac{n_1}{\sqrt{\pi}}\int_{-\infty}^{\infty} (v-v_1)\sqrt{\alpha}e^{-\alpha(v-v_1)^2}\,dv = \frac{n_1}{2\sqrt{\pi\alpha}}e^{-\alpha(v-v_1)^2}\Big|_{-\infty}^{\infty} = 0$$

The second integral is what we calculated in (a).

$$\frac{n_1}{\sqrt{\pi}}\int_{-\infty}^{\infty} v\sqrt{\alpha}e^{-\alpha(v-v_1)^2}\,dv = 0 + v_1\frac{n_1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\sqrt{\alpha}e^{-\alpha(v-v_1)^2}\,dv = v_1 n_1$$

Similarly for the second distribution function, we take $n_1 \to n_0$ and $v_1 \to -v_0$

$$\frac{n_0}{\sqrt{\pi}}\int_{-\infty}^{\infty} v\sqrt{\alpha}e^{-\alpha(v+v_0)^2}\,dv = -v_0 n_0$$

$$\int_{-\infty}^{\infty} vf(v)dv = \frac{n_0}{\sqrt{\pi}}\int_{-\infty}^{\infty} v\sqrt{\alpha}e^{-\alpha(v+v_0)^2}\,dv + \frac{n_1}{\sqrt{\pi}}\int_{-\infty}^{\infty} v\sqrt{\alpha}e^{-\alpha(v-v_1)^2}\,dv = n_1 v_1 - n_0 v_0$$

Plugging back in,

$$\boxed{\langle v\rangle = \frac{n_1 v_1 - n_0 v_0}{n_1 + n_0}} \qquad (10\ pts)$$

c.) The total current density for electrons is given by

$$J = q_e \int_{-\infty}^{\infty} vf(v)dv = -e(n_1 v_1 - n_0 v_0)$$

If the total current density is 0, then

$$\boxed{n_1 v_1 = n_0 v_0} \qquad (10\ pts)$$

Partial credit is generously given for attempts on this problem.