SOLUTIONS

UCLA Department of Electrical Engineering
EE2H – Physics for Electrical Engineers
Winter 2017
Midterm, February 9 2017, (1:45 minutes)

Name ______________________________ Student number______________________________

This is a closed book exam – you are allowed 1 page of notes (front+back).

Check to make sure your test booklet has all of its pages – both when you receive it and when you turn it in.

Remember – there are several questions, with varying levels of difficulty, be careful not to spend too much time on any one question to the exclusion of all others.

Exam grading: When grading, we focusing on evaluating your level of understanding, based on what you have written out for each problem. For that reason, you should make your work clear, and provide any necessary explanation. In many cases, a correct numerical answer with no explanation will not receive full credit, and a clearly explained solution with an incorrect numerical answer will receive close to full credit. CIRCLE YOUR FINAL ANSWER.

If an answer to a question depends on a result from a previous section that you are unsure of, be sure to write out as much of the solution as you can using symbols before plugging in any numbers, that way at least you will still receive the majority of credit for the problem, even if your previous answer was numerically incorrect.

Please be neat – we cannot grade what we cannot decipher.

	Topic	Max Points	Your points
Problem 1	Band diagram and electrical conduction	40	
Problem 2	Quantum Mechanics	40	
Problem 3	Density of States	20	
Total		100	

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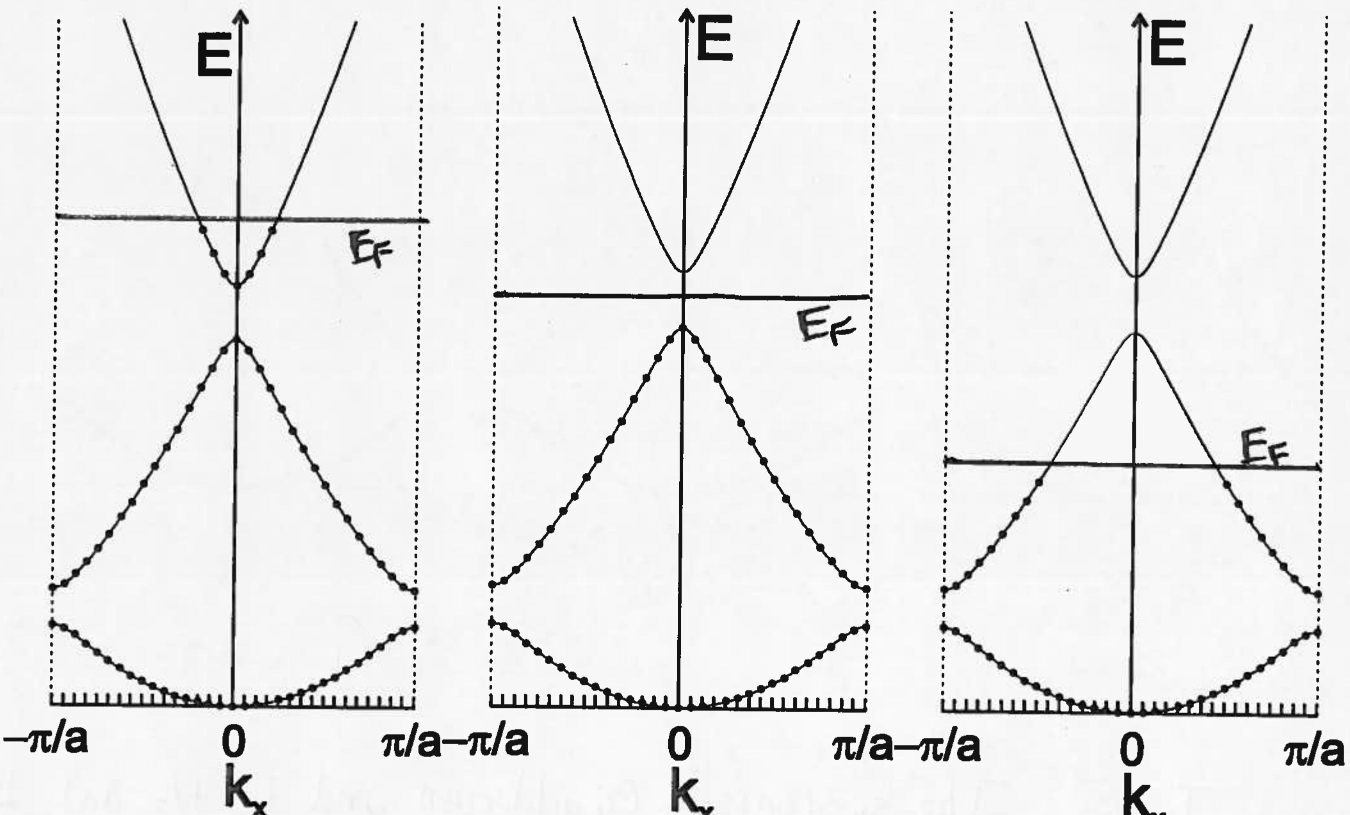
1. (40 points) Band diagram and electrical conduction

Consider the following band structure diagram for electrons in a 1D crystal with lattice constant *a*. Solid circles indicate occupied electron states. Empty circles in the valence band indicates an unoccupied electron state (i.e. a hole).

(a) (15 point) Consider the following band structure diagrams for a 1D crystal at temperature T=0 K. Sketch the Fermi level E_F on each diagram. For each diagram, is the material a conductor or an insulator?



Case A Case B Case C

At T=0

Case A

Conductor

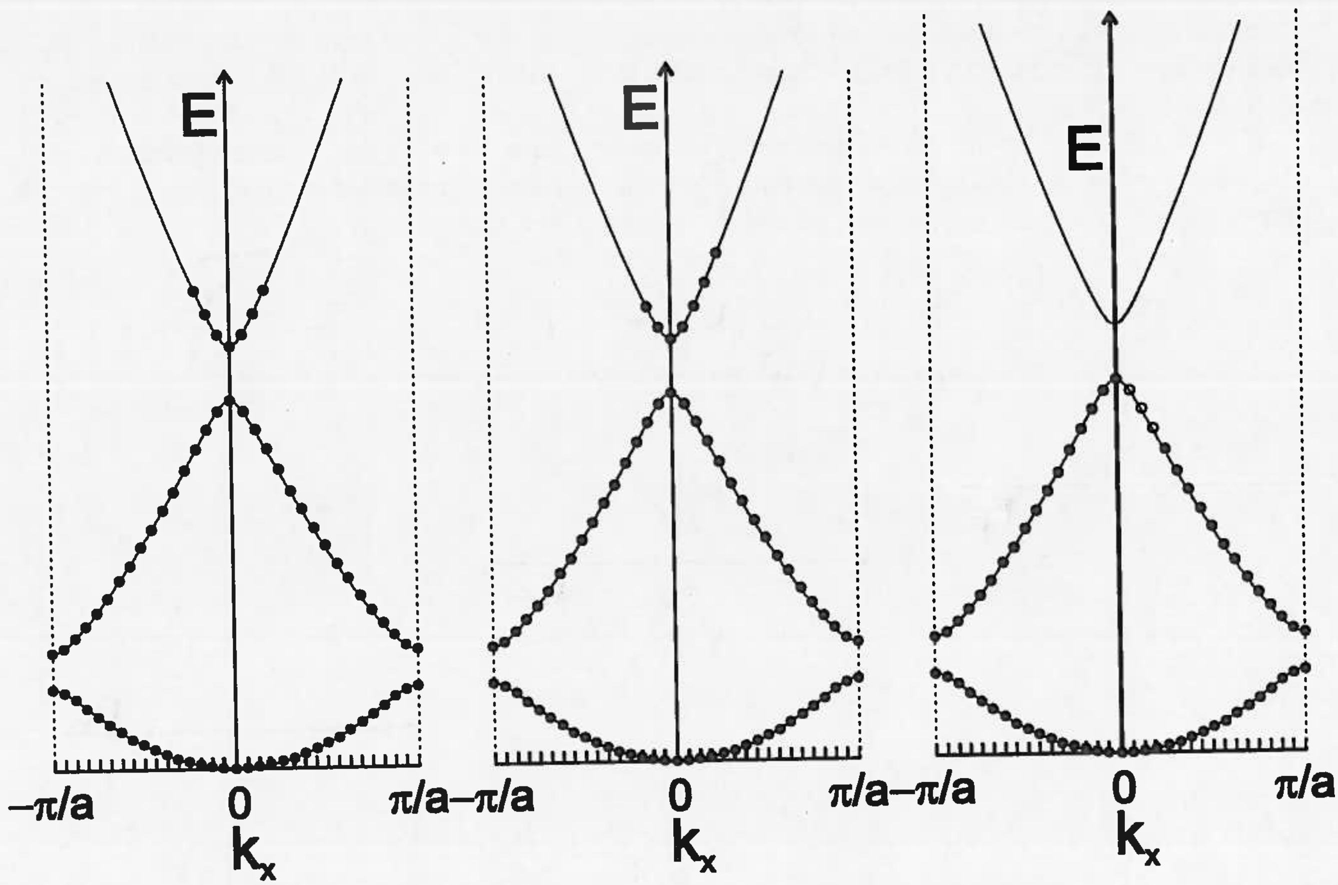
Case B

Insulator

Case C

Conductor

(b) (15 points) For each case shown below in the bandstructure diagrams, state whether the electrical current I flowing in the 1D crystal is zero, positive (+x direction), or negative (-x direction).



Case A Case B Case C

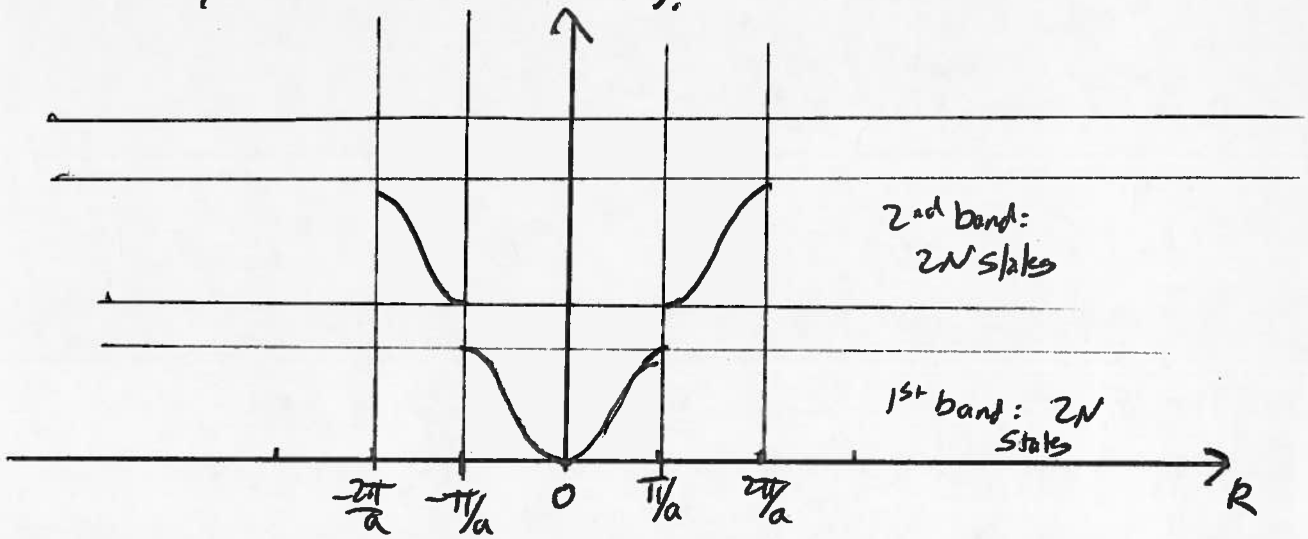
Case A $I = 0$. The system is in equilibrium and the $+k$ and $-k$ states are equally populated.

Case B $I < 0$. Nonequilibrium electron distribution in conduction band with more occupied states with $k > 0$. Since group velocity is positive for $k > 0$, $\langle v_x \rangle > 0$. Since electrons have negative charge $I < 0$.

Case C $I < 0$
Conduction band is empty, but there are empty states in valence band. Excess electron occupation in $k < 0$ states. However group velocity is positive for $k < 0$ states in valence band so $\langle v_x \rangle > 0$ and $I < 0$.

(c) (10 points) Explain why a 1D crystal (length L, lattice constant a) made up of a single species of atom can only be a semiconductor if each atom has an even number of valence electrons. While you may cite any necessary equations, you must clearly give an explanation in text.

~~The~~ In a crystal with lattice spacing a will have $2N$ states in each band where $N = L/a$ = # of atoms in crystal. The factor of 2 originates from the fact that two spin states exist for each state in k-space. According to Pauli exclusion principle, the electrons will fill up the available states (lowest energy states first at $T=0$).



If each atom contributes an even # of valence electrons, then the Fermi level will fall inside a band gap & the material will be a semiconductor/insulator.

If each atom contributes an odd # of valence electrons, then the Fermi level will fall inside a band, and the material will be a conductor.

2. Quantum Mechanics (40 points)

Consider a free electron system in a 1D crystal of length L with energy dispersion $E=\hbar^2k^2/2m^*$.

(a) (10 points) If the electron has a wavefunction $\psi(x,t=0)=\frac{1}{\sqrt{L}}e^{-jk_0x}$, what is the expectation value (i.e. average measured value) for the momentum p?

$$\boxed{\langle p\rangle = -\hbar k_0}$$

$$\langle p\rangle = -j\hbar\int_0^L \frac{1}{\sqrt{L}}e^{jk_0x}\frac{\partial}{\partial x}\frac{1}{\sqrt{L}}e^{-jk_0x}\,dx$$

$$= -j\hbar\frac{1}{L}\int_0^L(-jk_0)\,dx$$

$$= -\hbar k_0$$

(b) (10 points) Consider two electrons:

electron A has a wavefunction $\psi_A(x,t=0)=\frac{1}{\sqrt{a\sqrt{2\pi}}}e^{-x^2/4a^2}e^{jk_0x}$,

and electron B has a wavefunction $\psi_B(x,t=0)=\frac{1}{\sqrt{a\sqrt{2\pi}}}e^{-x^2/16a^2}e^{j4k_0x}$, where $a=10^{-9}$ m.

Which electron has a larger uncertainty in position? Which electron has a larger uncertainty in momentum?

$$|\psi_A(x)|^2=\frac{1}{a\sqrt{2\pi}}e^{-x^2/2a^2} \qquad |\psi_B(x)|^2=\frac{1}{a\sqrt{2\pi}}e^{-x^2/8a^2}=\frac{1}{a\sqrt{2\pi}}e^{-x^2/2(2a)^2}$$

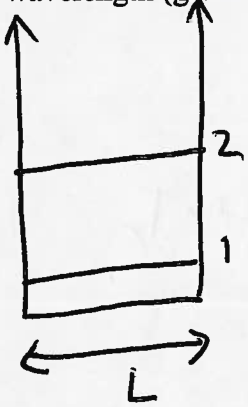
Uncertainty in A : $\Delta x = a$

Uncertainty in B $\Delta x = 2a$

B has larger position uncertainty.

so A must have larger momentum uncertainty $\Delta p\Delta x \geq \hbar/2$

(c) (10 points) Consider an electron (with effective mass $m^*=0.067m_0$) in a 1D GaAs quantum well of width L=1 nm with infinitely high barriers. If the electron is in state n=2, and makes a transition to state n=1 by emitting a photon, what will the energy (given in units eV) and wavelength (given in units μm) of the photon be?



$$E_n = \frac{\hbar^2 \pi^2 n^2}{2m^* L^2}$$

$$\hbar\omega = E_2 - E_1 = \frac{3}{2}\frac{\hbar^2\pi^2}{m^* L^2} = 2.7\times10^{-18}\,\text{J}$$

$$\boxed{\hbar\omega = h\nu = 16.9\,\text{eV}}$$

$$\nu = 4.07\times10^{15}\,\text{Hz}$$

$$\boxed{\lambda = \frac{c}{\nu} = 7.36\times10^{-8}\,\text{m} = 0.074\,\mu\text{m}}$$

(d) (10 points) Explain why an electron in a quantum well cannot have zero kinetic energy ($E_1\neq0$). Explain what this has to do with the uncertainty principle (using text, and any relevant equations if needed).

If the kinetic energy was zero, then the momentum would be zero with zero uncertainty.
Since the position uncertainty is finite ($\Delta x \approx L$) this would violate uncertainty principle.

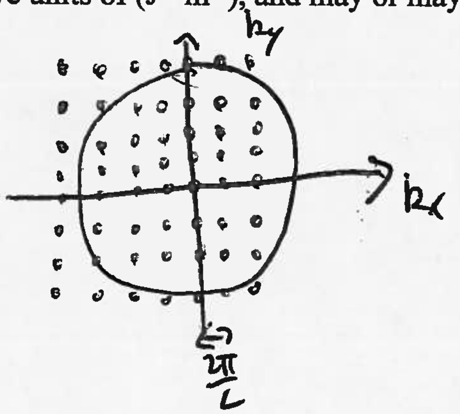
3. Density of States in a two-dimensional crystal (challenge problem) (20 points)

Consider a two dimensional free electron system of size $L \times L$ which contains free electron solutions of the form

$$\psi(x,y) = \frac{1}{L} e^{jk_x x} e^{jk_y y}, \text{ where } k_x = n_x \frac{2\pi}{L}, \text{ and } k_y = n_y \frac{2\pi}{L},$$

and $n_x = -\infty, \cdots, -1, 0, 1, \cdots, \infty$, and $n_y = -\infty, \cdots, -1, 0, 1, \cdots, \infty$, and $E = \frac{\hbar^2 k^2}{2m}$.

Calculate an expression for the 2D density of states $N(E)$ per unit energy for electrons. The expression should have units of (J^{-1} m^{-2}), and may or may not have explicit dependence on energy E.



Find # of states in circle of radius k.

$$\# = \underset{\text{spin}}{2} \times \underset{\text{area in k-space}}{\pi k^2} \times \underset{\text{2D density in k-space}}{\left(\frac{L}{2\pi}\right)^2} = \frac{2\pi k^2 L^2}{4\pi^2}$$

Convert to energy $E = \frac{\hbar^2 k^2}{2m}$ $k^2 = \frac{2mE}{\hbar^2}$

$$\# = \frac{L^2}{2\pi} \frac{2mE}{\hbar^2} = \frac{L^2 mE}{\pi \hbar^2}$$

$$\boxed{N(E) = \frac{1}{L^2}\frac{d\#}{dE} = \frac{m}{\pi \hbar^2}}$$

Fundamental constants

Planck's constant:	h=6.63×10^{-34} J s	h=4.14×10^{-15} eV s
	$\hbar$=$h/2\pi$ =1.06×10^{-34} J s	$\hbar$= $h/2\pi$ =6.58×10^{-16} eV s
Permittivity of free space	ϵ_0=8.85×10^{-12} F/m	ϵ_0=8.85×10^{-14} F/cm
Permeability of free space	μ_0=4π × 10^{-7} Ns^2/C^2	
Conversion from eV to J	1 eV=1.60×10^{-19} J	
Boltzmann's constant	k=1.38×10^{-23} J/K	k=8.62×10^{-5} eV/K
Bare electron mass	m_0=9.11×10^{-31} kg	
Speed of light	c=2.998x10^{8} m/s	c=2.998x10^{10} cm/s
Fundamental charge	e = 1.602×10^{-19} C	

1 Å=10^{-10} m, 1 nm=10^{-9} m, 1 μm=10^{-6} m.

Material properties

Silicon

All parameters at room temp	Silicon	GaAs
Crystal Structure	Diamond	Zincblende
a	5.43 Å	5.65 Å
Mass density	2.33 g/cm^3	5.31 g/cm^3
ϵ_r	11.8	13.2
E_g	1.11 eV	1.43 eV
μ_n	1350 cm^2/V s	8500 cm^2/V s
μ_p	480 cm^2/V s	400 cm^2/V s
m^*_n	0.26m_0	0.067m_0
m^*_p	0.49m_0	0.5m_0
Effective DOS N_c	2.8×10^{19} cm^{-3}	4.7×10^{17} cm^{-3}
Effective DOS N_v	1.0×10^{19} cm^{-3}	7.0×10^{18} cm^{-3}
n_i	1.5×10^{10} cm^{-3}	2×10^{6} cm^{-3}

Useful equations

Electron momentum: $p = mv = \hbar k = h/\lambda$ Planck relation: $E = hf = \hbar\omega$

Time independent Schrodinger's Equation (1D): $$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)=E\psi(x)$$

Expectation value for position $\langle x\rangle = \int_{-\infty}^{\infty} xP(x)dx = \int_{-\infty}^{\infty} x|\psi(x)|^2 dx$, $\langle x^2\rangle = \int_{-\infty}^{\infty} x^2|\psi(x)|^2 dx$.

Expectation value for momentum $\langle p\rangle = -j\hbar\int_{-\infty}^{\infty}\psi^*(x)\frac{\partial}{\partial x}\psi(x)dx = \cancel{\hbar\int_{-\infty}^{\infty} k|\psi(k)|^2 dk}$,

Uncertainty $$\Delta x = \sqrt{\left\langle\left(x-\langle x\rangle\right)^2\right\rangle} = \sqrt{\langle x^2\rangle - \langle x\rangle^2} \qquad \Delta p_x = \sqrt{\left\langle\left(p_x-\langle p_x\rangle\right)^2\right\rangle} = \sqrt{\langle p_x^2\rangle - \langle p_x\rangle^2}$$

Heisenberg uncertainty principle: $$(\Delta x)(\Delta p_x) \geq \frac{\hbar}{2}$$

Solution to 1D particle in box (infinite quantum well of width L with boundary condition $\psi(0)=\psi(L)=0$)

$$\psi(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right) \text{ for } 0<x<L,\ \ \psi(x)=0 \text{ otherwise} \quad , E_n=\frac{n^2\hbar^2\pi^2}{2mL^2},\ \ n=1,2,...$$

Solution to 3D free particle in volume L^3: (with periodic boundary conditions)

$$\psi(x,y,z)=\frac{1}{L^{3/2}}e^{j\mathbf{k}\cdot\mathbf{r}}\ ,\ E=\frac{\hbar^2k^2}{2m}\quad k_x=n_x\frac{2\pi}{L},k_y=n_y\frac{2\pi}{L},k_z=n_z\frac{2\pi}{L},\ \ n_x,n_y,n_z=...-2,-1,0,1,2,...$$

Ohm's law: $\mathbf{J}=\sigma\boldsymbol{\mathcal{E}}$

Conductivity of free electron gas (i.e. metal): $\sigma=\frac{ne^2\bar{t}}{m}$ $\qquad \sigma=\rho^{-1}$

Conductivity of semiconductor: $\sigma=ne\mu_n+pe\mu_p$

Semiconductor electron/hole mobility $\mu_n=\frac{e\bar{t}_n}{m_n^*}$ $\qquad \mu_p=\frac{e\bar{t}_p}{m_p^*}$

3D free electron Density of States $N(E)=\frac{1}{2\pi^2}\left(\frac{2m^*}{\hbar^2}\right)^{3/2}E^{1/2}$

Electron Effective mass $m^*=\frac{\hbar^2}{d^2E/dk^2}$

Fermi-Dirac distribution for electrons: $f(E)=\frac{1}{e^{(E-E_F)/kT}+1}$ and for holes: $f_h(E)=1-f(E)$

Equilibrium Carrier concentrations $n_0=\int_{E_C}^{\infty}f(E)N(E)dE$

Equilibrium Carrier concentrations in non-degenerate limit (E_C-$E_F \gg kT$ and E_F-$E_V \gg kT$).

$$n_0=N_Ce^{-(E_C-E_F)/kT}\quad ,\ p_0=N_Ve^{-(E_F-E_V)/kT}$$

$$N_C=2\left(\frac{2\pi m_n^*kT}{h^2}\right)^{3/2}\quad ,\ N_V=2\left(\frac{2\pi m_p^*kT}{h^2}\right)^{3/2}$$

$$n_i=\sqrt{n_0p_0}=\sqrt{N_CN_V}e^{-E_g/2kT}$$

Intrinsic Fermi Level $E_i=\frac{kT}{2}\ln\left(\frac{N_V}{N_C}\right)+\frac{E_V+E_C}{2}$

Einstein relation for diffusion coeff: $D=\frac{kT}{e}\mu$ $\qquad$ Diffusion length $L=\sqrt{D\tau}$

Debye screening length (for n-type): $L_D=\sqrt{\frac{\epsilon kT}{e^2n_0}}$

Dielectric relaxation time $\tau_D=\frac{\epsilon}{\sigma}=\frac{\epsilon}{n_0e\mu_n}$ (n-type) $=\frac{\epsilon}{p_0e\mu_p}$ (p-type)

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