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EE 1 F 2012

Midterm I Solution

Definition on Symbol:

① ⓐ $\vec{E}_n(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q_n(\vec{r}-\vec{r}_n)}{([\vec{r}-\vec{r}_n']\cdot[\vec{r}-\vec{r}_n'])^{3/2}}$, $\vec{r}_i = x_i\hat{x} + y_i\hat{y} + z_i\hat{z}$

$\vec{r}_1' = 4x - 3\hat{z}$

$= E_{nx}\hat{x} + E_{ny}\hat{y} + E_{nz}\hat{z}$; $\vec{r}_2' = 2x + \hat{z}$

$\vec{r}_a = 5\hat{x} + 6\hat{z}$

$\sum_{n=1,2} E_{nz}(\vec{r}_a) = 0$; Want Q_1, Given $Q_2 = 4nC$

$$0 = \cancel{\frac{1}{4\pi\varepsilon_0}}\left(Q_1 \frac{z_a - z_1}{((x_a-x_1)^2+(z_a-z_1)^2)^{3/2}} + Q_2 \frac{z_a - z_2}{((x_a-x_2)^2+(z_a-z_2)^2)^{3/2}}\right)$$

d_1 ↓ $C_1 = \frac{5}{(34)^{3/2}}$

d_2 ↓ $C_2 = \frac{9}{(82)^{3/2}}$

$$Q_1 = -Q_2 \, C_2/C_1 = \boxed{-8.32nC}$$

ⓑ $F_x = 0 \Rightarrow E_x = 0$ at $\vec{r}_a$:

$\sum_{n=1,2} E_{x_n} = 0 = E_x$

$$0 = \cancel{\frac{1}{4\pi\varepsilon_0}}\left(Q_1 \frac{x_a - x_1}{d_1 = 34^{3/2}} + Q_2 \frac{x_a - x_2}{d_2 = 82^{3/2}}\right)$$

$\to E_1 = 3/34^{3/2}$ $\to E_2 = 1/82^{3/2}$

$$Q_1 = -Q_2 \frac{E_2}{E_1}$$

$$= \boxed{-44.945nC}$$

② ⓐ $L = 5m$, $\rho_L = 12x^2 \left[mC/m^2\right]$

$$Q = \int_0^L \rho_L dx = 4x^3 \Big|_0^L = 4L^3 = \boxed{500\,mC}$$

ⓑ $\rho = 3$, $\rho_s(\rho, z) = \rho z^2\, nC/m^2$

$$Q = \iint \rho_s(\rho, z)\, ds, \quad ds = (\rho\, d\phi)\, dz$$

$$= \int_{z=0}^{z=4} \int_{\phi=0}^{\phi=2\pi} \rho z^2 \rho\, d\phi\, dz$$

$$= \rho^2 \int_0^4 z^2 dz \int_0^{2\pi} d\phi$$

$$= 9 \cdot 2\pi \left(\frac{z^3}{3} \Big|_0^4 \right) = 6\pi \cdot 64 = \boxed{384\pi\, nC}$$

$$= 1.20637\,\mu C$$

ⓒ $Q = \iiint \rho_v dv$; $r = 4m$, $\rho_v = \frac{10}{r\sin\theta} \left[\frac{C}{m^3}\right]$;

$$dv = dr\,(r\sin\theta\, d\phi)(r\, d\theta) = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{4m} \frac{10}{r\sin\theta} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 10(2\pi)(\pi)\left(\frac{r^2}{2}\Big|_0^4\right) = 10(2\pi)(\pi)\left(\frac{16}{2}\right)$$

$$= \boxed{160\pi^2\, C}$$

$$= 1579.14\, C$$

③ ⓓ $\rho_s = 6xy \ C/m^2$

$\iint \rho_s \, ds$; $ds = dy\,dx$ or $dx\,dy$

I choose $dy\,dx$.

$$\iint \rho_s \, dy\,dx = \int_0^2 \int_0^x \rho_s \, dy\,dx + \int_2^4 \int_0^{4-x} \rho_s \, dy\,dx$$

$$= \int_0^2 3xy^2 \Big|_{y=0}^{y=x} dx + \int_2^4 3xy^2 \Big|_{y=0}^{y=4-x} dx$$

$$= \int_0^2 3x^3 dx + \int_2^4 3x(4-x)^2 dx$$; Yow TI-89!

$$= \boxed{32C}$$

③ ⓐ $\dfrac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|\|\vec{B}\|} = \cos\theta_{AB}$;

$\vec{A}\cdot\vec{B} = (-3)(2)+(1)(-5)+(-2)(1)$

$= -6-5-2 = \boxed{-13}$

$\|\vec{A}\| = \sqrt{3^2+1^2+2^2} = \sqrt{14}$

$\|\vec{B}\| = \sqrt{2^2+5^2+1^2} = \sqrt{30}$

$\theta_{AB} = \arccos\left(\dfrac{-13}{\sqrt{14}\sqrt{30}}\right) = \boxed{2.2579}$ or 129.37°

ⓑ

$Proj_{\vec{C}}\vec{A} = \vec{A}\cdot\dfrac{\vec{C}}{\|\vec{C}\|}$

$\|\vec{C}\| = \sqrt{1+4^2} = \sqrt{17}$

$\vec{A}\cdot\vec{C} = 1\cdot1+(-2)(4) = 1+(-8) = -7$

$\vec{A}\cdot\dfrac{\vec{C}}{\|\vec{C}\|} = \boxed{\dfrac{-7}{\sqrt{17}}}$

Warning had the problem asked for the vector component along $\vec{C}$ then the answer is $\left(\vec{A}\cdot\dfrac{\vec{C}}{\|\vec{C}\|}\right)\dfrac{\vec{C}}{\|\vec{C}\|}$. In this problem, both shoulde be acceptable since no specification given in the problem.

3 c $\vec{A}\times\vec{B}\cdot\vec{C} = (\vec{A}\times\vec{B})\cdot\vec{C}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ -3 & 1 & -2 \\ 2 & -5 & 1 \end{vmatrix} = \hat{a}_x(1-10) - \hat{a}_y(-3+4) + \hat{a}_z(15-2)$$

$$= -9\hat{a}_x - \hat{a}_y + 13\hat{a}_z$$

$$\vec{A}\times\vec{B}\cdot\vec{C} = (-9\hat{a}_x - \hat{a}_y + 13\hat{a}_z)\cdot(\hat{a}_y + 4\hat{a}_z)$$

$$= -1 + 52 = \boxed{51}$$

④ ⓐ

$$Q = \iiint \rho_v \, dv \; ; \; dv = d\rho\,(\rho\, d\phi)\, dz = \rho\, d\rho\, d\phi\, dz$$

$$Q = \int_0^1 \int_0^{\pi/4} \int_0^2 4\rho^2 \cos\phi \, z \, \rho \, d\rho\, d\phi\, dz$$

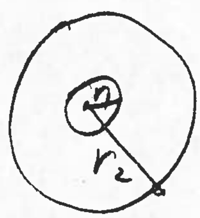
$$= \left(\frac{z^2}{2}\Big|_0^1\right) \int_0^{\pi/4} 4\cos\phi \, d\phi \int_0^2 \rho^3 \, d\rho$$

$$= \left(\frac{1}{2}\right)(\cancel{4})\left(\sin\phi \Big|_0^{\pi/4}\right) \frac{\rho^4}{\cancel{4}}\Big|_0^2$$

$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\left(2^4\right) = \boxed{4\sqrt{2}\ nC}$$

$$= 5.657\ nC$$

ⓑ



$r_1 = 2cm, r_2 = 4cm$, $r_1 = 0.02m$, $r_2 = 0.04m$

$\rho_v = 5 mC/m^3$

$$Q = \iiint \rho_v \, dv \; , \; dv = dr\,(r\sin\theta\, d\phi)(r\, d\theta)$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_{r_1}^{r_2} (5mC)\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 5mC\,(2\pi) \int_0^{\pi} \sin\theta \, d\theta \int_{r_1}^{r_2} r^2 \, dr$$

$$= (5mC)(2\pi)(2)\left(\frac{r^3}{3}\Big|_{r_1}^{r_2}\right)$$

(4)(b) cont'd

$$Q = \frac{20}{3}\pi\left(\underset{\overset{\shortparallel}{0.04\text{m}}}{r_2^3} - \underset{\overset{\shortparallel}{0.02\text{m}}}{r_1^3}\right) \text{mC}$$

$$= \boxed{373\pi \text{nC}}$$

$$= 1.1729 \mu\text{C}$$

5 a $Q_{eq} = Q_1 + Q_2$; $\rho_{s_1} = 8\,nC/m^2$, $\rho_{s_2} = -6\,nC/m^2$

$r_1 = 1m$, $r_2 = 2m$

$$Q = \iint \rho_s \, ds \; ; \; ds = (r\sin\theta d\phi)(r d\theta) = r^2 \sin\theta d\theta d\phi$$

$$Q_1 = \int_0^{2\pi}\int_0^{\pi} \rho_{s_1} r_1^2 \sin\theta d\theta d\phi = \underbrace{\rho_{s_1}}_{8\left[\frac{nC}{m^2}\right]} \underbrace{r_1^2}_{1[m^2]} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi} \sin\theta d\theta}_{2}$$

$$= \boxed{32\pi\, nC}$$

$$Q_2 = \int_0^{2\pi}\int_0^{\pi} \rho_{s_2} r_2^2 \sin\theta d\theta d\phi = \underbrace{\rho_{s_2}}_{-6\left[\frac{nC}{m^2}\right]} \underbrace{r_2^2}_{4m^2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{2\pi} \sin\theta d\theta}_{2}$$

$$= -96\pi\, nC$$

$$Q_{eq} = -64\pi\, nC$$

$$\oiint \vec{D}\cdot d\vec{s} = Q_{eq} \; ; \; d\vec{s} = \hat{a}_r (r\sin\theta d\phi)(r d\theta)$$

Think about it yourself or come ask me in person

- $\vec{D}(r,\theta,\phi) = D_r(r,\theta,\phi)\hat{a}_r + D_\theta(r,\theta,\phi)\hat{a}_\theta + D_\phi(r,\theta,\phi)\hat{a}_\phi$
- By symm. argument $D_\theta = D_\phi = 0$, $D_r(r,\theta,\phi) = D_r(r)$

$$\vec{D}(r,\theta,\phi) = D_r(r)\hat{a}_r = \vec{D}(r)$$

8 a contd:

$$Q_{eq} = \int_0^{2\pi}\int_0^{\pi} D_r(r)\, r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r(r)\, r^2 \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2}$$

$$Q_{eq} = 4\pi r^2 D_r(r)$$

$$D_r(r) = \frac{Q_{eq}}{4\pi r^2} = \frac{-64\pi}{4\pi r^2} [nC]$$

$$\vec{D}(r) = D_r(r)\hat{a}_r = \frac{-64\pi}{4\pi r^2} \hat{a}_r [nC]$$

$$\vec{D}(3) = \frac{-64\pi}{36\pi} \hat{a}_r \left[\frac{nC}{m^2}\right]$$

$$= \boxed{\frac{-16}{9} \left[\frac{nC}{m^2}\right]}$$

$$= 1.7778 \left[\frac{nC}{m^2}\right]$$

(5) (b) $\vec{D}(\vec{r}) = \frac{1}{4\pi}\int \frac{\rho_l(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')\,dl'$

$\vec{r} = 3\hat{x}$; $\rho_l(\vec{r}') = 5\mu C/m$.

$\vec{r}' = y'\hat{y} + z'\hat{z}$; We know that $y'^2 + z'^2 = 4$

Alter cylindrical coordinate so that $\rho' = \sqrt{y'^2+z'^2} = 2$ and $\hat{\rho}'$ is in yz-plane.

Then $\vec{r}' = 2\hat{\rho}'$.

The rest is easy...

$\vec{r}-\vec{r}' = 3\hat{x} - 2\hat{\rho}'$; $|\vec{r}-\vec{r}'| = \sqrt{(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')}$

x and my <u>new</u> $\hat{\rho}'$ are perpendicular so—

$|\vec{r}-\vec{r}'| = \sqrt{3^2+2^2} = \sqrt{13}\,m$ $\quad dl' = 2\,d\phi'[m]$; ϕ changed accordingly to yz-plane

$$\vec{D}(3,0,0) = \frac{1}{4\pi}\int_0^{2\pi} \frac{5\mu C/m}{13^{3/2}m^3}(3\hat{x}-2\hat{\rho}')(2\,d\phi'[m])$$

$$= \frac{5\cdot 2}{4\pi(13)^{3/2}}\left[\int_0^{2\pi} 3\hat{x}\,d\phi' - \int_0^{2\pi} 2\hat{\rho}'\,d\phi\right]\left[\frac{\mu C}{m^2}\right]$$

$$= \left(\frac{5}{2\pi(13)^{3/2}}\right)\left[\hat{x}(6\pi) - \int_0^{2\pi} 2\hat{\rho}'\,d\phi'\right]\left[\frac{\mu C}{m^2}\right]$$

5 6 Cont'd

$$\hat{\rho}' = \cos\phi'\,\hat{x} + \sin\phi'\,\hat{y}$$

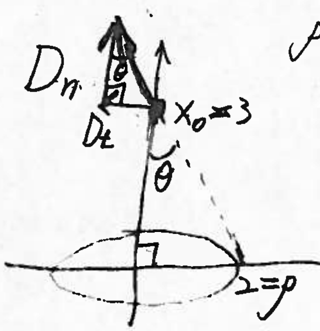
$$= \left(\frac{5}{2\pi(13)^{3/2}}\right)\left[\hat{x}(6\pi) - 2\underbrace{\int_0^{2\pi}[\cos\phi'\,\hat{x} + \sin\phi'\,\hat{y}]\,d\phi'}_{0}\right]\left[\frac{\mu C}{m^2}\right]$$

$$= \boxed{\frac{15}{(13)^{3/2}}\,\hat{x}\left[\frac{\mu C}{m^2}\right]}$$

$$= 0.320\,\hat{x}\left[\frac{\mu C}{m^2}\right] = \vec{D}(3,0,0)$$

Method 2: By geometry:

$$\vec{D} = \hat{x}\,\frac{1}{4\pi}\int \frac{\rho_\ell \cos\theta}{\underbrace{2^2+3^2}_{\rho^2 + x_0^2}}\,d\ell\,; \quad d\ell = \rho\, d\phi = 2\,d\phi$$



$D_{normals}$ adds while $D_{tangential}$ cancels.
That's why $\cos\theta$, since vector sum.

$$\cos\theta = \frac{3}{\sqrt{2^2+3^2}} = \frac{3}{\sqrt{13}}$$

$$\vec{D} = \hat{x}\,\frac{\rho_\ell}{2\pi}\left(\frac{3}{13^{3/2}}\right)\int_0^{2\pi} d\phi = \hat{x}\,\frac{3\rho_\ell}{13^{3/2}}\left[\frac{\mu C}{m^2}\right]$$

$$\boxed{\vec{D}(3,0,0) = 0.320\left[\frac{\mu C}{m^2}\right]}$$

⑥ $\oiint \vec{D}(\rho,\phi,z)\cdot d\vec{s} = \int_{l_1}^{l_2} \rho_L \, dl = Q$; $\rho_L = 10\,nC/m$

$d\vec{s} = \hat{\rho}\,(\rho\, d\phi)\, dz$, $dl = dz$

By symmetry argument:

$\vec{D}(\rho,\phi,z) = D_\rho(\rho)\,\hat{\rho} = \vec{D}(\rho)$

$$Q = \int_{l_1}^{l_2} \rho_L \, dz = \rho_L (l_2 - l_1)$$

$$Q = \int_{l_1}^{l_2}\int_0^{2\pi} D_\rho(\rho)\cdot \rho\, d\phi\, dz$$

$$Q = D_\rho(\rho)\,\rho \int_{l_1}^{l_2} dz \int_0^{2\pi} d\phi$$

$$= D_\rho(\rho)\,\rho\,(l_2 - l_1)(2\pi) = \rho_L (l_2 - l_1)$$

$$D_\rho(\rho) = \frac{\rho_L \cancel{(l_2 - l_1)}}{2\pi\rho \cancel{(l_2 - l_1)}} = \frac{\rho_L}{2\pi\rho}$$

$\vec{E}(\rho) = \dfrac{\rho_L}{2\pi\varepsilon_0\rho}\hat{\rho}$; $\rho = \sqrt{3^2 + 4^2} = 5$

$$\vec{E}(5) = \frac{10\,[nC/m]}{2\pi\varepsilon_0(5)}\hat{\rho} = \boxed{\frac{1}{\pi\varepsilon_0}\left[nV/m\right]}$$

$[F/m]$

$$= 3.5967\times 10^{10}\, nV/m = \boxed{35.967\ V/m}$$

7 ⓐ $\vec{r}_1' = \hat{x} + 3\hat{z}$, $\vec{r}_2' = -2\hat{x} + \hat{y} + 5\hat{z}$

$\vec{r}_p = \hat{x} - 2\hat{y} + 3\hat{z}$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{|\vec{r}_p - \vec{r}_1'|} + \frac{1}{4\pi\varepsilon_0}\frac{Q_2}{|\vec{r}_p - \vec{r}_2'|}$$

$\vec{r}_p - \vec{r}_1' = -2\hat{y}$, $\vec{r}_p - \vec{r}_2' = 3\hat{x} - 3\hat{y} - 2\hat{z}$

$|\vec{r}_p - \vec{r}_1'| = 2$, $|\vec{r}_p - \vec{r}_2'| = \sqrt{9+9+4} = \sqrt{22}$

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{2nC}{2} + \frac{-4nC}{\sqrt{22}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\left(1 - \frac{4}{\sqrt{22}}\right)nC\right] = \frac{0.1472}{4\pi(8.85\times10^{-12})}[nV]$$

$$= 1.3235\times10^{9}\,nV = \boxed{1.3235\,V}$$

(7)(b) $Q = \iiint \rho_v \, dv$; $dv = dr(r\sin\theta\, d\phi)(r\, d\theta) = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$\rho_v = \rho_0 (1 - (r/a))^2$

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho_0 (1-(r/a))^2 r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \rho_0 \int_0^{2\pi} d\phi \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{2} \int_0^{a} r^2\left(1 - \frac{2r}{a} + \frac{r^2}{a^2}\right) dr$$

$$= \rho_0 (2\pi)(2) \int_0^{a} r^2 - \frac{2}{a} r^3 + \frac{1}{a^2} r^4 \, dr$$

$$= 4\pi\rho_0 \left[\frac{r^3}{3} - \frac{1}{2a} r^4 + \frac{1}{5a^2} r^5\right]\Big|_0^a$$

$$= 4\pi\rho_0 \left[\frac{a^3}{3} - \frac{a^3}{2} + \frac{a^3}{5}\right]$$

$$= 4\pi\rho_0 \left[1/30\right] = \boxed{\frac{2\pi\rho_0}{15} a^3}$$

For $r > a$:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{a}_r = \frac{1}{4\pi\varepsilon_0} \frac{(2\pi\rho_0 a^3/15)}{r^2} \hat{a}_r$$

$$= \boxed{\frac{\rho_0 a^3}{30 r^2} \hat{a}_r}$$

7) b) $r > a$

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} + C$$

$$V = \frac{\rho_0 a^3}{3\epsilon_0 r} + C$$; Let's choose $\lim_{r\to\infty} V = 0$

$\therefore 0 = C$

$$\boxed{V = \frac{\rho_0 a^3}{3\epsilon_0 r}}$$