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EE1 Midterm1

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* $\varepsilon_0 = \frac{1}{36\pi} \times 10^{-9}$ F/m

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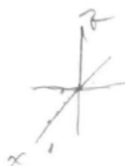
1 Point charges $Q_1$ and $Q_2$ are respectively located at $(4,0,-3)$ and $(2,0,1)$. If $Q_2$ is $4\,nC$ find $Q_1$ such that

a) The electric field intensity has no $z$ component at point $(5,0,6)$

b) The force on a unit test charge has no $x$-component at point $(5,0,6)$



a) $\vec{E} = \frac{Q_1}{4\pi\varepsilon_0} \frac{(\hat{a}_x + 9\hat{a}_z)}{(9^2+1^2)^{3/2}} + \frac{Q_2}{4\pi\varepsilon_0} \frac{(3\hat{a}_x + 5\hat{a}_z)}{(3^2+5^2)^{3/2}}$

$r_p - r_1 = (1,0,9)$

$r_p - r_2 = (3,0,5)$

$\vec{E}_z = \frac{1}{4\pi\varepsilon_0}\left(\frac{9Q_1}{(82)^{3/2}} + \frac{5Q_2}{(34)^{3/2}}\right)\hat{a}_z = 0$

$\frac{9Q_1}{(82)^{3/2}} = -\frac{5Q_2}{(34)^{3/2}} \rightarrow Q_1 = \frac{-5Q_2(82)^{3/2}}{9(34)^{3/2}} = \boxed{-8.323\,nC}$

b) $F = qE \rightarrow F_x = qE_x$

$\vec{E}_x = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{(82)^{3/2}} + \frac{3Q_2}{(34)^{3/2}}\right)\hat{a}_x \rightarrow \vec{F}_x = 0 = \frac{q}{4\pi\varepsilon_0}\left(\frac{Q_1}{(82)^{3/2}} + \frac{3Q_2}{(34)^{3/2}}\right)\hat{a}_x$

$\frac{Q_1}{(82)^{3/2}} = -\frac{3Q_2}{(34)^{3/2}} \rightarrow Q_1 = \frac{-3Q_2(82)^{3/2}}{(34)^{3/2}} = \boxed{-44.945\,nC}$

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2 Determine the total charge

(a) On line $0 < x < 5\,m$ if $\rho_L = 12x^2$ mC/m

$$q_{tot} = \int_L \rho_L \, dL = \int_0^5 (12x^2 \cdot 10^{-3})\,dx$$

$$= 4\cdot 10^{-3}\left[x^3\right]_0^5 = 4\cdot 10^{-3}(5^3) = \boxed{0.5\,C}$$

(b) On a cylinder $\rho = 3$, $0 < z < 4\,m$ if $\rho_s = \rho z^2$ nC/m$^2$

$$q_{tot} = \iint \rho_s \, \rho \, d\phi \, dz = \int_0^4\int_0^{2\pi} (\rho z^2)\rho \, d\phi \, dz \cdot 10^{-9}$$

$$= 2\pi\rho^2\left[\tfrac{1}{3}z^3\right]_0^4 \cdot 10^{-9} = \boxed{1.206\,\mu C}$$

$\rho = 3$

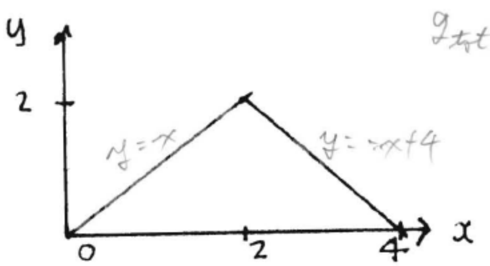
(c) Within the sphere $r = 4\,m$ if $\rho_v = \dfrac{10}{r\sin\theta}$ C/m$^3$.

$$q_{tot} = \int_0^{2\pi}\int_0^{\pi}\int_0^4 \left(\frac{10}{r\sin\theta}\right) r^2 \sin\theta \, dr\, d\theta\, d\phi = 2\pi \int_0^{\pi}\int_0^4 10\, r \, dr\, d\theta$$

$$= 20\pi^2\left[\tfrac{1}{2}r^2\right]_0^4 = 10\pi^2 \cdot 16 = \boxed{1579\,C}$$

(d) How much charge is enclosed in the triangular region if $\rho_s = 6xy$ C/m$^2$

$$q_{tot} = \int_S \rho_s \, dS = \iint 6xy \, dy\, dx$$




$$q_{tot} = \int_0^2\int_0^x 6xy\,dy\,dx + \int_2^4\int_0^{-x+4} 6xy\,dy\,dx$$

$$= 3\int_0^2 x\cdot x^2\,dx + 3\int_2^4 x\cdot(4-x)^2\,dx$$

$$= \boxed{32\,C}$$

3 Let $\underline{A} = -3\,\underline{a}_x + \underline{a}_y - 2\,\underline{a}_z$ $= (-3, 1, -2)$

$\underline{B} = 2\,\underline{a}_x - 5\,\underline{a}_y + a_z$ $= (2, -5, 1)$

$\underline{C} = \underline{a}_y + 4\,\underline{a}_z$. $= (0, 1, 4)$

determine

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5 a) Smaller of the two angles between $\underline{A}$ and $\underline{B}$

$\vec{a}\cdot\vec{b} = |a||b|\cos\theta \rightarrow \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|a||b|}\right)$

$\theta / \phi$

$\theta = \cos^{-1}\left(\frac{-6-5-2}{\sqrt{14}\cdot\sqrt{30}}\right) = 129.37°$

$\phi = 180 - \theta = \boxed{50.63°}$

0 b) The vector component of $\underline{A}$ along $\underline{C}$

$(\vec{a}\cdot\vec{c})\,\hat{a}_c = (0+1-8)\left(\frac{\hat{a}_y + 4\hat{a}_z}{\sqrt{17}}\right) = \frac{-7}{\sqrt{17}}(\hat{a}_y + 4\hat{a}_z)$

$= \frac{-7}{\sqrt{17}}\hat{a}_y - \frac{28}{\sqrt{17}}\hat{a}_z = \boxed{-1.698\,\hat{a}_y - 6.791\,\hat{a}_z}$

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0 c) $(\underline{A} \times \underline{B} \cdot C)$

$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ -3 & 1 & -2 \\ 2 & -5 & 1 \end{vmatrix} \cdot C$

b) $(\vec{a}\cdot\hat{a}_c)\,\hat{a}_c$ · take answer, divide by $\sqrt{17}$

c) 51

$\langle -9, -1, 13\rangle \cdot \langle 0, 1, 4\rangle = 51$

4 a A volume charge density $\rho_v = 4\rho^2 z \cos\phi$ exists inside a wedge defined by

$nC/m^3$

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$0 < \rho < 2$

$0 < \phi < \pi/4$

$0 < z < 1$

How much charge is contained inside the wedge?

$$Q_{tot} = \int_0^1 \int_0^{\pi/4} \int_0^2 (4\rho^2 z \cos\phi)\rho \, d\rho \, d\phi \, dz \cdot 10^{-9}$$

$$= 4\int_0^1 z \, dz \cdot \int_0^{\pi/4} \cos\phi \, d\phi \cdot \int_0^2 \rho^3 d\rho \cdot 10^{-9}$$

$$= \boxed{5.657 nC}$$

4 b A spherical shell extending from $r = 2$ cm to $r = 4$ cm has a uniform charge density $\rho_v = 5$ $mC/m^3$. How much charge is contained?

$$Q_{tot} = \int_0^{2\pi} \int_0^{\pi} \int_{0.02}^{0.04} 5 r^2 \sin\theta \, dr \, d\theta \, d\phi \cdot 10^{-3}$$

$$= 5 \cdot 2\pi \cdot \int_0^{\pi} \sin\theta \cdot \int_{0.02}^{0.04} r^2 dr \cdot 10^{-3}$$

$$= \boxed{1.173 \mu C}$$

5 If spherical surfaces $r=1m$ and $r=2m$ respectively carry uniform surface charge densities of $8\ nC/m^2$ and $-6\ nC/m^2$, find $\underline{D}$ at $r=3\ m$.

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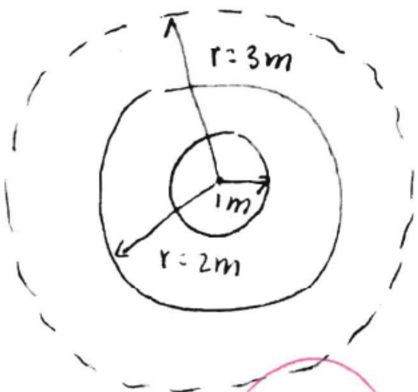


$\oint D\cdot dS = q_{tot}$

$4\pi r_3^2 D = q_{tot} = \rho_{s_1} S_1 + \rho_{s_2} S_2$

$4\pi r_3^2 D = (8\times10^{-9})4\pi r_1^2 + (-6\times10^{-9})4\pi r_2^2$

$\vec{D} = \boxed{-1.778\ nC/m^2\ \hat{a}_\rho}$

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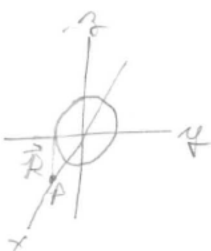
b) A ring placed along $y^2+z^2=4$, $x=0$ carries a uniform charge of $5\ \mu C/m$.

Find $\underline{D}$ at $P(3,0,0)$   $\vec{D} \parallel \hat{a}_x$

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$dQ = \rho_l\, dL = \rho_l \rho\, d\phi$

$d\vec{E} = \dfrac{dQ}{4\pi\varepsilon_0 |R|^2}\hat{a}_R$   $\vec{R} = \vec{R}_P - \vec{R}_{ring} = x\hat{a}_x - \rho\hat{a}_\rho$   $\hat{a}_R = \dfrac{\vec{R}}{|R|}$

$E = \int d\vec{E} = \int_0^{2\pi} \dfrac{\rho_l \rho\, d\phi(-\rho\hat{a}_\rho + x\hat{a}_x)}{4\pi\varepsilon_0 (\rho^2+x^2)^{3/2}} = \int_0^{2\pi} \dfrac{\rho_l \rho\, d\phi\, x}{4\pi\varepsilon_0(\rho^2+x^2)^{3/2}}\hat{a}_x$

6 An infinitely long but thin wire carries a line charge density of $\rho_L$ C/m and lies along the z axis

Use Gauss' law to calculate an expression for the electric field $\underline{E}(\rho, \phi, z)$.

If $\rho_L = 10$ nC/m what is the electric field at $(3, 4, 5)$?

$D = \varepsilon_0 \mathcal{E}$

$\vec{\mathcal{E}} = \vec{\mathcal{E}}(\rho)$, since due to symmetry, the $\phi$- and $z$- components cancel out

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$$\oint D \cdot d\mathcal{S} = \oint \varepsilon_0 \mathcal{E} \cdot d\mathcal{S} = Q_{tot} = \int \rho_L \, d\mathcal{L}$$

$$\int_0^{L}\int_0^{2\pi} \varepsilon_0 \mathcal{E} \rho \, d\phi \, dz = \int_0^{L} \rho_L \, dz$$

$$\varepsilon_0 \rho \mathcal{E} \cdot 2\pi \cdot \int_0^{L} dz = \rho_L \int_0^{L} dz$$

$$\mathcal{E} \cdot 2\pi \varepsilon_0 \rho = \rho_L$$

$$\boxed{\vec{\mathcal{E}}(\rho, \phi, z) = \frac{\rho_L}{2\pi\varepsilon_0\rho} \hat{a}_\rho}$$

$$\cancel{\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ}$$

$\varepsilon_0 = \frac{1}{36\pi} \times 10^{-9}$

$$\rho = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5$$

$$\vec{\mathcal{E}}(\rho = 5) = \frac{10 \cdot 10^{-9}}{2\pi\varepsilon_0 5} = \boxed{36 \text{ V/m}}$$

Question 7 (a) Two point charges $Q_1 = 2\,nC$ and $Q_2 = -4\,nC$ are located at $(1,0,3)$ and $(-2,1,5)$ respectively. Determine the potential at $P(1,-2,3)$.

$$V = -\int_0^a \mathcal{E} \cdot d\mathcal{L} \qquad r_p - r$$

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?

(b) Given that a spherically symmetric charge distribution is given by

$$\rho_v = \rho_0 \left(1 - (r/a)\right)^2 \quad \text{for} \quad r \le a$$
$$= 0 \qquad\qquad r \ge a$$

Find $\underline{E}$ and $V$ for $r \ge a$.

$$\oint D \cdot ds = \int_v \rho_v \, dV$$

$$4\pi r^2 \varepsilon_0 \mathcal{E} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \left(1 - \frac{r}{a}\right)^2 r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$2\cancel{4\pi} r^2 \varepsilon_0 \mathcal{E} = \cancel{2\pi} \int_0^{\pi} \sin\theta \, d\theta \int_0^{a} \left(r^2 - \frac{r^3}{a}\right) dr$$

$$\cancel{2} r^2 \varepsilon_0 \mathcal{E} = \cancel{2} \left[\frac{1}{3} r^3 - \frac{1}{4}\frac{r^4}{a}\right]_0^a = \frac{1}{3}a^3 - \frac{1}{4}\frac{a^4}{a} = \frac{1}{12}a^3$$

$$\boxed{\vec{\mathcal{E}}(r) = \frac{a^3}{12\varepsilon_0 r^2} \hat{a}_r}$$

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