Fall 2009 Midterm 1 Solutions Prof. C. Joshi

1 i) $\underline{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\, \underline{a}_R$ (N)

ii) $\underline{E} = \dfrac{Q}{4\pi\varepsilon_0 R^2}\, \underline{a}_R$ (V/m)

iii) $\underline{D} = \varepsilon_0 \underline{E}$ (C/m²)

iv) $\oint_S \underline{D}\cdot d\underline{s} = Q_{enclosed}$ (C)

v) $\underline{\nabla}\cdot\underline{D} = \rho_v$ C/m³

vi) $\oint_S \underline{D}\cdot d\underline{s} = \int_v (\underline{\nabla}\cdot\underline{D})\, dv$ C

OK V_{21} or $V_2 - V_1$

vii) $V_{12} = -\int_1^2 \underline{E}\cdot d\underline{l}$ (V)

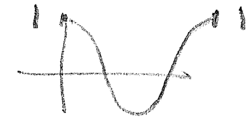
viii) $\underline{E} = -\underline{\nabla} V$ (V/m)

ix) $d\underline{s} = \rho\, d\phi\, dz\, \underline{a}_\rho$ m²

x) $d\underline{V} = \rho\, d\phi\, d\rho\, dz\, \underline{a}_z$. m³

$\int$ w.r.t. ϕ first both terms give

$$-\underbrace{\left[\cos\phi\right]_0^{2\pi}}_{0} \rho^2 d\rho \quad - \quad z\rho \underbrace{\left[\cos\phi\right]_0^{2\pi}}_{0} = 0$$



Solution 2

a) $\underline{D} = z \sin\phi \; \underline{a}_\rho + z\cos\phi \; \underline{a}_\phi + \rho \sin\phi \; \underline{a}_z \quad C/m^2$

$\boxed{20 min}$

$\underline{\nabla} \cdot \underline{D} = \rho_v$

In cylindrical co-ordinate system

$$\underline{\nabla} \cdot \underline{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho D_\rho\right) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

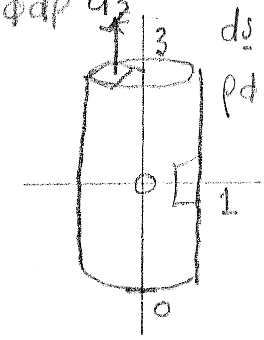
$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \; z\sin\phi\right) + \frac{1}{\rho}\frac{\partial}{\partial \phi}\left(z\cos\phi\right) + \frac{\partial}{\partial z}\left(\rho\sin\phi\right)$$

$$= \frac{z\sin\phi}{\rho} - \frac{z\sin\phi}{\rho} + 0$$

$$= 0$$

$d\underline{s} = \rho d\phi d\rho \; \underline{a}_z$

$d\underline{s} = \rho d\phi dz \; \underline{a}_\rho$



b) Divergence theorem

$$\oint_S \left(\underline{D} \cdot d\underline{s}\right) = \int_V \left(\underline{\nabla} \cdot \underline{D}\right) dv$$

but RHS = 0 because $(\underline{\nabla} \cdot \underline{D}) = 0$ from (a)

so LHS must be = 0.

Check Using a cylindrical gaussian surface

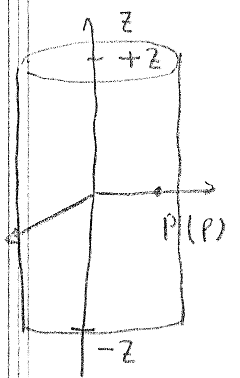
$$\oint \underline{D} \cdot d\underline{s} = \pm 2 \int_0^{2\pi} d\phi \int_0^1 D_z \; \rho \, d\phi \, d\rho + \int_0^3 \int_0^{2\pi} D_\rho \; \rho \, d\phi \, dz$$

$$= \qquad \rho\sin\phi \; \rho \, d\phi \, d\rho \qquad\qquad z \sin\phi \; \rho \, d\phi \, dz .$$

30 min

Solution to 3

a)



Gauss' law $\oint_S \underline{D} \cdot d\underline{s} = Q_{encl} = \int_{-z}^{+z} \rho_L \, dl$

$= 2z \rho_L$.

Since $\rho << 2z$ no contribution to the closed integral from the top or bottom

$$\underline{D} = D_\rho \underline{a}_\rho + D_z \underline{a}_z \simeq D_\rho \underline{a}_\rho$$

$$d\underline{s} = \rho d\phi d\rho \, \underline{a}_z + \rho d\phi dz' \, \underline{a}_\rho$$

$$\oint \underline{D} \cdot d\underline{s} = \int_{-z}^{+z} \int_0^{2\pi} D_\rho \, \rho \, d\phi \, dz' = 2\pi \rho (2z) D_\rho$$

$$2\pi \rho \cancel{(2z)} \varepsilon_0 E_\rho = \cancel{2z} \rho_L .$$

$$E_\rho = \rho_L / 2\pi\varepsilon_0 \rho \quad \text{V/m} .$$

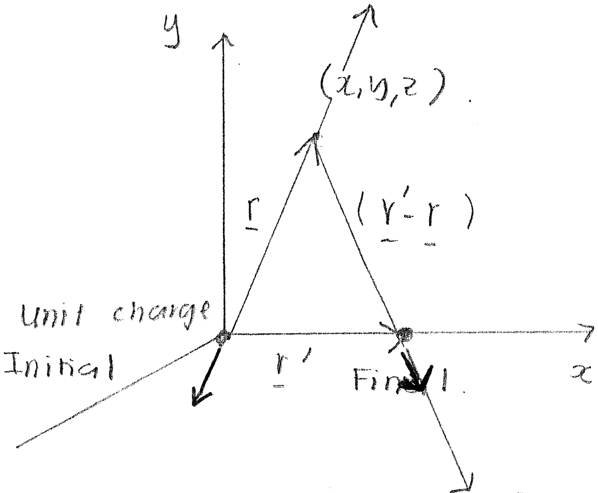
b) For a uniform line charge $\underline{E}_\rho = \frac{\rho_L}{2\pi\varepsilon_0} \frac{\underline{r}_p}{|\underline{r}_p|^2}$

$$\underline{r}_p = (-3, 2, 6) - (-3, +2, +4) = (0, 0, 2) = 2 \underline{a}_z .$$

$$|\underline{r}_p|^2 = 4$$

$$\underline{E}_\rho = \frac{10 \times 10^{-9}}{\cancel{2}\pi\varepsilon_0} \frac{\cancel{2} \underline{a}_z}{4} = \frac{2.5 \times 10^{-9}}{\pi \varepsilon_0} \underline{a}_z \; (\text{V/m})$$

Solution to 4.



$$\underline{F} = \frac{Q}{4\pi\varepsilon_0} \frac{a_R}{|\underline{r}'-\underline{r}|^2}$$

With the unit charge at the origin (0,0,0)

(1) $$a_R = \frac{-x\, a_x - y\, a_y - \cancel{z\, a_z}}{(x^2+y^2+\cancel{z^2})^{1/2}} = 0.5\, a_x - 0.5\sqrt{3}\, a_y$$

The force is in the direction of a_R

Since z component of the force is zero. z must be zero.

When the unit charge is moved to (1,0,0).

(2) $$a_R = \frac{(1-x)\, a_x - y\, a_y - \cancel{z\, a_z}}{\{(1-x)^2 + y^2 + \cancel{z^2}\}^{1/2}} = 0.6\, \underline{a}_x - 0.8\, \underline{a}_y.$$

From equation (1)

$$\frac{-y}{(x^2+y^2)^{1/2}} = -0.5\sqrt{3}\quad , \quad \frac{-x}{(x^2+y^2)^{1/2}} = 0.5.$$

$\therefore\ x = -y/\sqrt{3}$. or $+y = -\sqrt{3}\,x$.

(2) $\Rightarrow$ $$\frac{(1-x)}{-y} = -0.6/0.8.$$

Solve after substitution

$(1-x)/\sqrt{3}\,x = -3/4.$

$1-x = -0.75\sqrt{3}\,x$

$x = 1/(1-0.75\sqrt{3})$

$y = -\sqrt{3}\,x\ ,\ z = 0$

(bonus)

Solution to 5.

The potential V from a line charge is given by

$$V(\rho) = -\int E_\rho \, d\rho = -\int \frac{\rho_L}{2\pi\varepsilon_0 \rho} d\rho$$

$$= -\frac{\rho_L}{2\pi\varepsilon_0}\left[\ln \rho\right] + \text{constant}.$$

If we have two line charges, we use principle of linear superposition

$$V_{total} = -\frac{\rho_L}{2\pi\varepsilon_0}\left\{\ln \rho_1 + \ln \rho_2\right\} + \text{constant}.$$

but at the origin $(0,0,0)$ $\rho_1 = (1+0+4)^{1/2} = \sqrt{5}$

$\rho_2 = (1+4+0)^{1/2} = \sqrt{5}$

$$100\,V = \left[-\frac{4\times10^{-9}}{2\pi\varepsilon_0}\left\{2\ln(\sqrt{5})\right\}\right] + \text{const}.$$

$\therefore$ Const $= 100V + [\quad]$

Now $V(4,1,3)$ is calculated by calculating

$$\rho_1'(4,1,3) = \left((4-1)^2 + \underbrace{(1-1)^2}_{\text{for all } y} + (3-2)^2\right)^{1/2} = \sqrt{10}$$

$$\rho_2' = (4,1,3) = \left((4+1)^2 + (1-2)^2 + \underbrace{(3-3)^2}_{\text{for all } z}\right) = \sqrt{26}$$

solution to 5 cont.

$$V(4,1,3) = -\frac{4\times10^{-9}}{2\pi\epsilon_0}\left\{\ln\left(\sqrt{10}\right)+\ln\left(\sqrt{26}\right)\right\}$$

$$+ \text{ const}$$

substitute for const to get $V(4,1,3)$.

20 min

Solution to 6

x_1	y_1	$y = mx + c$
2	11	$= 4x + 3$.
x_2	y_2	$dy = 4\,dx$
4	19	

Work done $W = -Q \int \underline{E} \cdot d\underline{L}$

$$= -10^{-11} \int_{\text{initial}}^{\text{final}} (y\,a_x + x\,a_y) \cdot (dx\,a_x + dy\,a_y + dz\,a_z)$$

$$W = -10^{-11} \int_2^4 (4x+3)\,dx + 4x\,dx.$$

$$= -10^{-11} \int_2^4 (8x+3)\,dx = -10^{-11} \left[(4x^2 + 3x) \right]_2^4$$

$$= -10^{-11} \left[64 - 16 + 12 - 6 \right]$$

$$= -54 \times 10^{-11} \text{ J}$$

$$= -(0.54 \text{ nJ})$$