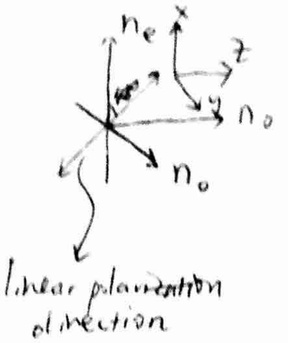
ECE 170A Midterm Winter 2018 Solution

Q1.




The waveplate should be in the $n_e$-$n_o$ plane

The wave is travelling along one of the ordinary axis. ($z$-axis)

polarization state varies periodically along $z$-axis with period $\frac{2\pi}{|k_e - k_o|}$

To have circular polarization, the optical wave must have linear polarization direction 45° to x and y axis. And the waveplate thickness is

$$l = \frac{1}{4}\cdot\frac{2\pi}{|k_e - k_o|} = \frac{\lambda}{4|n_e - n_o|} \quad \text{(quarter wave plate)}$$

use $n_o = 1.54423$, $n_e = 1.55332$

$$\Rightarrow l \approx 1.65 \times 10^{-2}\ \text{mm}$$

Q2. Kramers - Kronig relations

$$\chi'(\omega) = \frac{2}{\pi} P \int_0^\infty \frac{\omega' \chi''(\omega')}{\omega'^2 - \omega^2} d\omega', \quad \chi''(\omega) = -\frac{2}{\pi} P \int_0^\infty \frac{\omega \chi'(\omega')}{\omega'^2 - \omega^2} d\omega'$$

The formula indicates that the real and imaginary parts of $\chi(\omega)$ are not independent of each other, i.e. once we know one, we know the other.

Q3. With $n_1 = 1.54$, $n_2 = 1.47$, $n_3 = 1.00$, $d = 1.5\,\mu m$

$$V = \frac{2\pi}{\lambda} d \sqrt{n_1^2 - n_2^2} = \frac{4.326}{\lambda}$$ , where $\lambda$ is in $\mu m$

$$a_E = \frac{n_2^2 - n_3^2}{n_1^2 - n_2^2} = 5.57 \qquad a_M = \frac{n_1^4}{n_3^4}\frac{n_2^2 - n_3^2}{n_1^2 - n_2^2} = 81$$

For waveguide to support $TE_0$ mode but not $TM_0$ mode:

$$M_{TE} = 1 \text{ and } M_{TM} = 0 \Rightarrow \tan^{-1}\sqrt{a_E} < V < \tan^{-1}\sqrt{a_M}$$

$$\Rightarrow 1.168 < \frac{4.326}{\lambda} < 1.393$$

$$\Rightarrow 3.106\,\mu m < \lambda < 3.704\,\mu m$$

For waveguide to support $TE_0$ mode but not $TE_1$ mode:

$$M_{TE} = 1 \Rightarrow 0 < \frac{V}{\pi} - \frac{1}{\pi}\tan^{-1}\sqrt{a_E} \leq 1$$

$$\Rightarrow \tan^{-1}\sqrt{a_E} < V < \pi + \tan^{-1}\sqrt{a_E}$$

$$\Rightarrow 1.168 < \frac{4.326}{\lambda} < 4.310$$

$$\Rightarrow 1.004\,\mu m < \lambda < 3.704\,\mu m$$

For waveguide to support $TM_0$ mode but not $TM_1$ mode:

$$M_{TM} = 1 \Rightarrow 0 < \frac{V}{\pi} - \frac{1}{\pi}\tan^{-1}\sqrt{a_M} \leq 1$$

$$\Rightarrow \tan^{-1}\sqrt{a_M} < V < \pi + \tan^{-1}\sqrt{a_M}$$

$$\Rightarrow 1.393 < \frac{4.326}{\lambda} < 4.535$$

$$= 0.954\,\mu m < \lambda < 3.106\,\mu m$$

Q4. (a) Two-mode coupling

For general case, the coupled-mode equations are

$$\pm \frac{dA_\nu}{dz} = \sum_\mu i K_{\nu\mu} A_\mu e^{i(\beta_\mu - \beta_\nu)z}$$

where $K_{\nu\mu} = \omega \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\mathcal{E}}_\nu^* \cdot \Delta\epsilon \cdot \hat{\mathcal{E}}_\mu dx dy$ is the coupling coefficient between mode $\nu$ and mode $\mu$.

For two-mode coupling, there are only two modes (a and b)

So the coupled-mode equations are reduced to:

$$\pm \frac{dA}{dz} = iK_{aa}A + iK_{ab}Be^{i(\beta_b - \beta_a)z}$$

$$\pm \frac{dB}{dz} = iK_{bb}B + iK_{ba}Ae^{i(\beta_a - \beta_b)z}$$

use normal-mode expansion

$$A(z) = \tilde{A}(z)\exp[\pm i\int_0^z K_{aa}(z)dz]$$

$$B(z) = \tilde{B}(z)\exp[\pm i\int_0^z K_{bb}(z)dz]$$

The coupled-mode equations can be transformed to

$$\pm \frac{d\tilde{A}}{dz} = iK_{ab}(z)\tilde{B}e^{i\phi(z)} = iK_{ab}(z)\tilde{B}e^{i2\delta z}$$

$$\pm \frac{d\tilde{B}}{dz} = iK_{ba}(z)\tilde{A}e^{-i\phi(z)} = iK_{ba}(z)\tilde{A}e^{-i2\delta z}$$

where $\phi(z) = [\beta_b z \pm \int_0^z K_{bb}(z)dz] - [\beta_a z \pm \int_0^z K_{aa}(z)dz]$

For uniform perturbation, $\Delta\epsilon$ is only a function of x and y but not z

Then $K_{aa}$, $K_{bb}$, $K_{ab}$, $K_{ba}$ above are constants that are independent of z

So $A(z) = \tilde{A}(z)e^{\pm iK_{aa}z}$

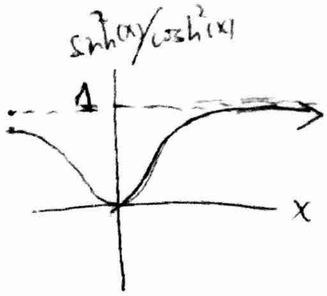
$B(z) = \tilde{B}(z)e^{\pm iK_{bb}z}$

and $2\delta z = \phi(z) = [(\beta_b \pm K_{bb}) - (\beta_a \pm K_{aa})]$

$2\delta$ is called the phase mismatch between two modes

Q4 (b). For contradirectional coupler, the coupling efficiency

$$\eta = \frac{K_{ba}^*}{K_{ab}} \frac{\sinh^2 \alpha_c l}{\cosh^2 \alpha_c l - \delta^2/K_{ab}K_{ba}}, \quad \text{where } \alpha_c = (K_{ab}K_{ba} - \delta^2)^{1/2}$$




For small $\delta$ : $\delta^2 < K_{ab}K_{ba}$

$\alpha_c$ will be real and positive

Then $\sinh \alpha_c l$ and $\cosh \alpha_c l$ are both monotonic functions with $\sinh \alpha_c l \to 1$ and $\cosh \alpha_c l \to 1$ as $l \to \infty$

So $\eta_{max} = \frac{K_{ba}^*}{K_{ab}^*}$ → maximum efficiency for contradirectional coupling when $\delta^2 < K_{ab}K_{ba}$

If $\eta = \frac{1}{2}\eta_{max}$, then

$$\frac{\sinh^2 \alpha_c l}{\cosh^2 \alpha_c l - \delta^2/K_{ab}K_{ba}} = \frac{1}{2}$$

$$\Rightarrow \frac{\sinh^2 \alpha_c l}{\cosh^2 \alpha_c l} \approx \frac{1}{2}$$

$$\Rightarrow \alpha_c l \approx \tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow l \approx \frac{1}{\alpha_c} \tanh^{-1}\left(\frac{1}{\sqrt{2}}\right), \quad \alpha_c = (K_{ab}K_{ba} - \delta^2)^{1/2}$$