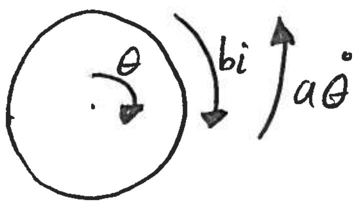
Midterm solutions EE 141 Winte 2014

1. KVL for the circuit: $v = Ri + L\frac{di}{dt} + e$

$$\Longrightarrow \frac{di}{dt} = \frac{v - Ri - e}{L} \quad (1)$$

2. 

$$J\ddot{\theta} = bi - a\dot{\theta} \quad (2)$$

3. $e = b\dot{\theta} \overset{(1)}{\Longrightarrow} \frac{di}{dt} = \frac{v - Ri - b\dot{\theta}}{L} \Longrightarrow s\,I(s) = \frac{V - RI - bs\Theta(s)}{L}$

$$I(R + Ls) = V - bs\Theta(s) \Longrightarrow I = \frac{V - bs\Theta}{R + Ls}$$

$$\overset{(2)}{\Longrightarrow} s^2 J\Theta = bI - as\Theta \Longrightarrow s^2 J\Theta = b \cdot \frac{V - bs\Theta}{R + Ls} - as\Theta$$

$$\Longrightarrow \Theta(s)\left[(R + Ls)s^2 J + as(R + Ls) + b^2 s\right] = bV$$

$$\Longrightarrow \frac{\Theta(s)}{V(s)} = \frac{b}{LJs^3 + (RJ + aL)s^2 + (aR + b^2)s}$$

4. $\dfrac{\theta}{V} = \dfrac{1}{8s^3 + 8s^2 + 3s}$

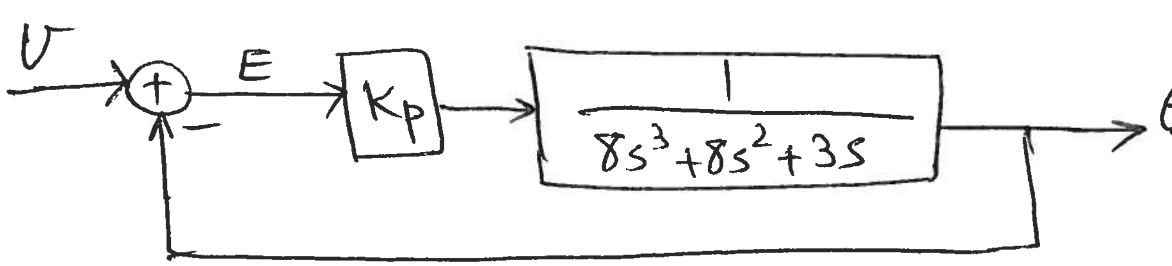
A necessary condition for stability is that all the coefficients of the characteristic polynomial be positive

$8s^3 + 8s^2 + 3s + 0s^0 \rightarrow$ not positive $\Rightarrow$ <u>not stable</u>

Another approach would be to consider that system has one pole at origin and two poles in the LHP. Nonrepeated poles on $j\omega$ axis makes the system <u>neutrally stable</u>.

From either approach, the system is $\boxed{\text{not stable}}$.

5.



Characteristic function of closed-loop system:

$$8s^3 + 8s^2 + 3s + K_p = 0$$

We'll use Routh's criterion.

$$
\begin{array}{c|cc}
s^3 & 8 & 3 \\
s^2 & 8 & K_p \\
s^1 & \frac{8\times 3 - 8\times K_p}{8} = 3 - K_p & 0 \\
s^0 & K_p &
\end{array}
$$

In order to have a stable system: $\begin{cases} 3 - K_p > 0 \\ K_p > 0 \end{cases}$

$\Rightarrow$ $\boxed{0 < K_p < 3}$ will make the system stable.

6. Characteristic function should have the same roots as:

$$(s+0.5)(s+0.25+0.25j)(s+0.25-0.25j) = (s+0.5)\left((s+0.25)^2+\frac{1}{16}\right) = 0$$

$$(s+0.5)\left(s^2+\frac{s}{2}+\frac{1}{16}+\frac{1}{16}\right) = (s+0.5)\left(s^2+\frac{s}{2}+\frac{1}{8}\right) = 0$$

$$8(s+0.5)\left(s^2+\frac{s}{2}+\frac{1}{8}\right) = (s+0.5)(8s^2+4s+1) = 0$$

$$8s^3+8s^2+3s+0.5 = 0$$

By comparing this to characteristic function in Q5,

K_p is equal to 0.5. $\Rightarrow$ $\boxed{K_p = 0.5}$

7. $$\frac{E}{V} = \frac{8s^3+8s^2+3s}{8s^3+8s^2+3s+0.5}$$

We need to find the steady-state error to the step input V

$$V=\frac{1}{s}\ ; \quad \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s.\frac{8s^3+8s^2+3s}{8s^3+8s^2+3s+0.5} \times \frac{1}{s}$$

$$\cancel{\frac{0}{0.5}} = 0$$

Yes! The system will be able to track step inputs with ~~constant error of 2~~ zero steady state error.

8. $$\frac{\Theta(s)}{V(s)} = \frac{0.5}{8s^3+8s^2+3s+0.5} = \frac{0.5}{8(s+0.5)\left((s+0.25)^2+\frac{1}{16}\right)}$$

where we used our knowledge about roots from Q6.

$$V(s)=\frac{1}{s} \Rightarrow \Theta(s) = \frac{0.5}{8(s+0.5)\left((s+0.25)^2+\frac{1}{16}\right)} \times \frac{1}{s}$$

$$= \frac{a}{s} + \frac{b}{s+0.5} + \frac{cs+d}{\left(s+\frac{1}{4}\right)^2+\frac{1}{16}}$$

$$a = s\,\Theta(s)\Big|_{s=0} = \frac{0.5}{8 \times 0.5 \times \frac{1}{8}} = 1$$

$$b = (s+0.5)\,\Theta(s)\Big|_{s=-0.5} = \frac{0.5}{8 \times \frac{1}{8} \times -0.5} = -1$$

$$\left(s+\frac{1}{4}\right)^2 + \frac{1}{16} = s^2 + \frac{s}{2} + \frac{1}{8}$$

$$\Rightarrow a(s+0.5)\left(s^2+\frac{s}{2}+\frac{1}{8}\right) + bs\left(s^2+\frac{s}{2}+\frac{1}{8}\right) + (cs+d)\,s(s+0.5) = \frac{0.5}{8}$$

Since $a = 1$ & $b = -1 \Rightarrow c = 0$ (There is no 3rd order term of s)

$$\left(s^2+\frac{s}{2}+\frac{1}{8}\right)\left((s+0.5)-s\right) + ds(s+0.5) = \frac{0.5}{8} \Rightarrow d = -0.5$$

$$\Rightarrow \Theta(s) = \frac{1}{s} - \frac{1}{s+0.5} - \frac{0.5}{\left(s+\frac{1}{4}\right)^2 + \frac{1}{16}}$$

$$\Rightarrow \theta(t) = \left[1 - e^{-0.5t} - 2e^{-\frac{1}{4}t}\sin\frac{t}{4}\right]u(t)$$

When $t \to \infty \Rightarrow \theta(t) \to 1$ so the e_{ss} to step input is zero.

9. $$\frac{\Theta(s)}{V(s)} = \frac{0.5}{8(s+0.5)\left(s^2+\frac{s}{2}+\frac{1}{8}\right)}$$

$$\simeq \frac{1}{8\left(s^2+\frac{s}{2}+\frac{1}{8}\right)}$$

[DC gain of the approximated system should be the same as original one]

$$= \frac{1/8}{s^2+\frac{s}{2}+\frac{1}{8}}$$

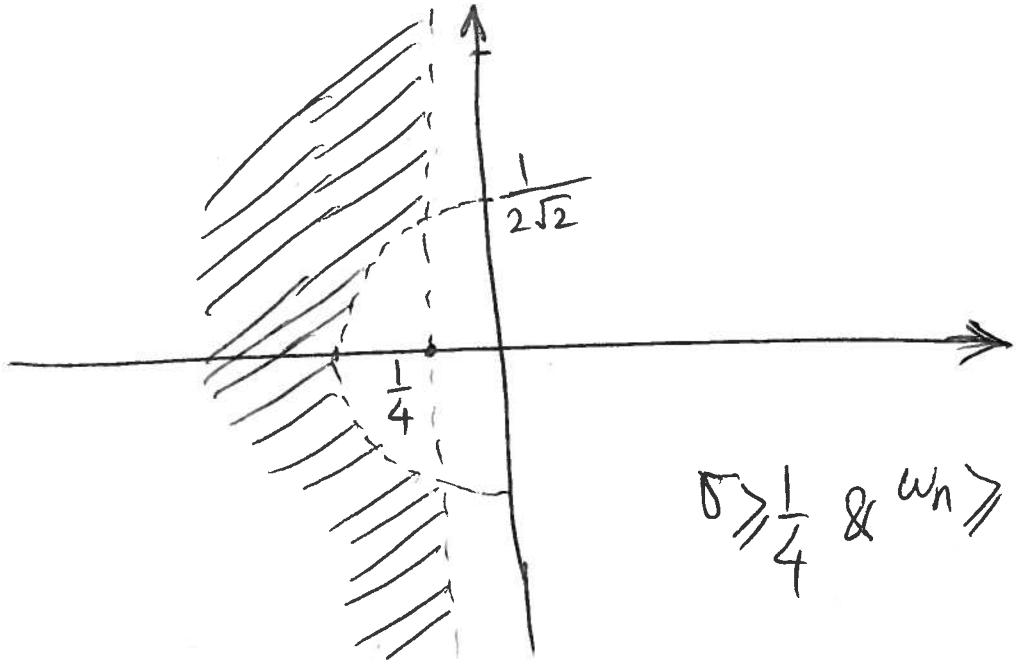
$$\Longrightarrow \zeta\omega_n = \frac{1}{4}, \quad \omega_n^2 = \frac{1}{8} \Longrightarrow \quad \omega_n = \frac{1}{2\sqrt{2}}, \quad \zeta = \frac{\sqrt{2}}{2}$$

$$t_r \simeq \frac{1.8}{\omega_n} = 1.8 \times 2\sqrt{2} = 5.0918 \text{ sec}$$

$$t_s = \frac{4.6}{\zeta\omega_n} = 4.6 \times 4 = 18.4 \text{ sec}$$

10. $$t_r \leq \frac{1.8}{\omega_n} \Rightarrow \omega_n \geq \frac{1.8}{t_r} \Rightarrow \omega_n \geq \frac{1}{2\sqrt{2}}$$

$$t_s \leq \frac{4.6}{\zeta\omega_n} = \frac{4.6}{\sigma} \Longrightarrow \sigma \geq \frac{1}{4}$$



$\sigma \geq \frac{1}{4}$ & $\omega_n \geq \frac{1}{2\sqrt{2}}$

11. Let's first start from ess for a ramp input:

By substituting $V = \frac{1}{s^2}$ in the ess equation of Q7

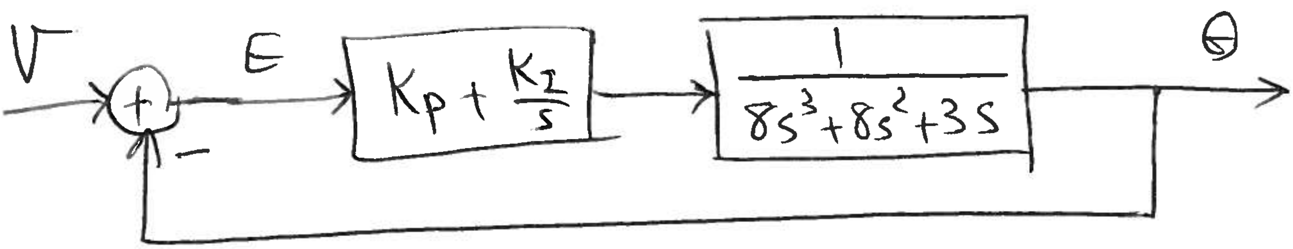
we have $\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = s \cdot \frac{8s^3+8s^2+3s}{8s^3+8s^2+3s+k_p} \cdot \frac{1}{s^2}$

$= \frac{3}{K_p}$

So just a proportional controller is not enough to make the steady-state error to the ramp input equal to zero

Let's choose more complicated controller:

If we have another "s" in the nominator of $sE(s)$, we will get the zero error. By changing the controller we can't chang the nominator of $E(s)$ but we can change its denominator. So if we can add a $\frac{1}{s}$ term in controller, the effect will be adding $\frac{1}{s}$ in the denominator of $E(s)$ and equally adding anothe s in the nominator of $E(s)$. So the controller term that we need to add is $\frac{K_I}{s} \Rightarrow D = K_p + \frac{K_I}{s}$



$$E(s) = \frac{1}{1 + \left(K_p + \frac{K_I}{s}\right)\frac{1}{8s^3+8s^2+3s}} V(s) = \frac{8s^4+8s^3+3s^2}{8s^4+8s^3+3s^2+K_p s+K_I} V(s)$$

$$V(s) = \frac{1}{s^2} \Rightarrow \lim_{s\to 0} sE(s) = s \cdot \frac{8s^4+8s^3+3s^2}{8s^4+8s^3+3s^2+K_p s+K_I} \cdot \frac{1}{s^2} = 0$$

So fortunately the e_{ss} in now zero!

Now we need to find suitable K_P & K_I that makes the system stable.

$$8s^4 + 8s^3 + 3s^2 + K_P s + K_I = 0$$

Routh array:

$$\begin{array}{c|ccc} s^4 & 8 & 3 & K_I \\ s^3 & 8 & K_P & \\ s^2 & 3-K_P & K_I & \\ s^1 & \frac{(3-K_P)K_P - 8K_I}{3-K_P} & & \\ s^0 & K_I & & \end{array}$$

$3 - K_P > 0 \Rightarrow K_P < 3$

$K_I > 0$

$\Rightarrow \dfrac{(3-K_P)K_P - 8K_I}{3-K_P} > 0$

$\Rightarrow (3-K_P)K_P - 8K_I > 0$

$\Rightarrow (3-K_P)K_P > 8K_I$

If we pick $K_P = 1 \Rightarrow (3-1)1 > 8K_I \Rightarrow K_I < \frac{1}{4}$

the we can pick $K_I = \frac{1}{8}$

With $\boxed{K_P = 1 \ \& \ K_I = \frac{1}{8}}$ all conditions are satisfied.

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