EE 141 – Midterm
Winter 2013

02/13/13

Duration: 1 hour and 40 minutes

The midterm is closed book and closed lecture notes. No calculators.
You can use a single page of handwritten notes.
Please carefully justify all your answers.

Problem 1: Consider two rooms in a house as depicted in Figure 1 . The temperature in the left room is denoted by x_1 and the temperature in the right room is denoted by x_2. The left room is equipped with a heater and we are interested in controlling the heater to regulate the temperature x_2.

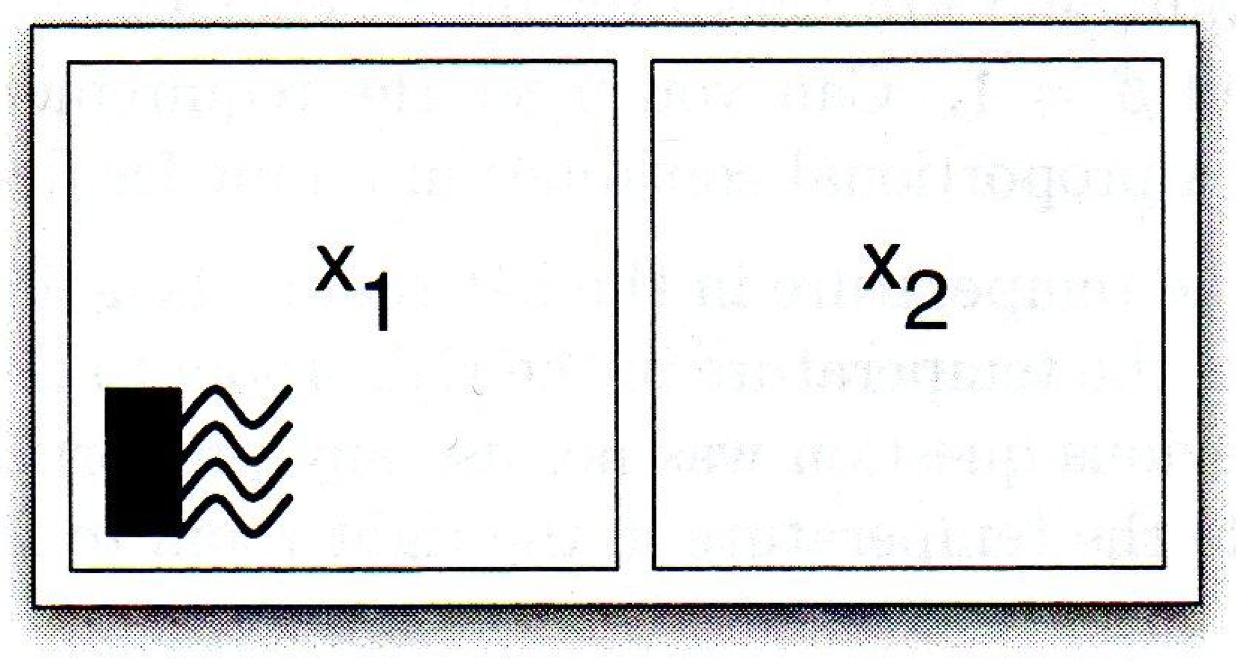


Figure 1: Two rooms in a house. The room in the left is equipped with a heater.

We assume that the evolution of x_1 is described by the following differential equation:

$$\frac{d}{dt}x_1 = -\alpha x_1 - \beta(x_1 - x_2) + u \tag{1}$$

where the term $-\alpha x_1$ describes heat lost to the exterior of the building, the term $-\beta(x_1 - x_2)$ represents the heat lost to the room in the right, and u represents the heat generated by the heater. The parameters α and β describe the rate at which heat is transferred to the exterior and to the room in the right, respectively.

1. Knowing that the room in the right does not lose heat to the exterior of the building but loses heat to the room in the left at a rate σ, and knowing that there is no other loss of heat, what is the differential equation describing the evolution of x_2?

2. Since our objective is to regulate x_2, compute the transfer function from u to x_2 assuming zero initial conditions.

3. Assume now that the physical model is defined by the following equations.

$$\frac{d}{dt}x_1 = -\alpha x_1 - x_2 + u \tag{2}$$

$$\frac{d}{dt}x_2 = \beta x_1 - \alpha x_2 \tag{3}$$

   (a) Compute the transfer function from u to x_2 assuming zero initial conditions.

   (b) Assuming that α and β are positive, which values of α and β would you chose to satisfy the following two requirements:

      i. rise time no greater than 0.45 seconds;

      ii. settling time no greater than 9.2 seconds;

   (c) The values of α and β depend on the materials used to construct the exterior and interior walls and are thus difficult to change. In this problem we consider that $\alpha = 1$ and $\beta = 1$. Can you meet the requirements of the previous question by designing a proportional controller in a unit feedback loop?

   (d) What is the temperature in the left room when the proportional controller is used to regulate the temperature in the right room to 20 degrees Celsius? (If the answer to the previous question was no, use any proportional controller that can be used to regulate the temperature in the right room to 20 degrees Celsius).

Problem 2: Consider the block diagram in Figure 2.

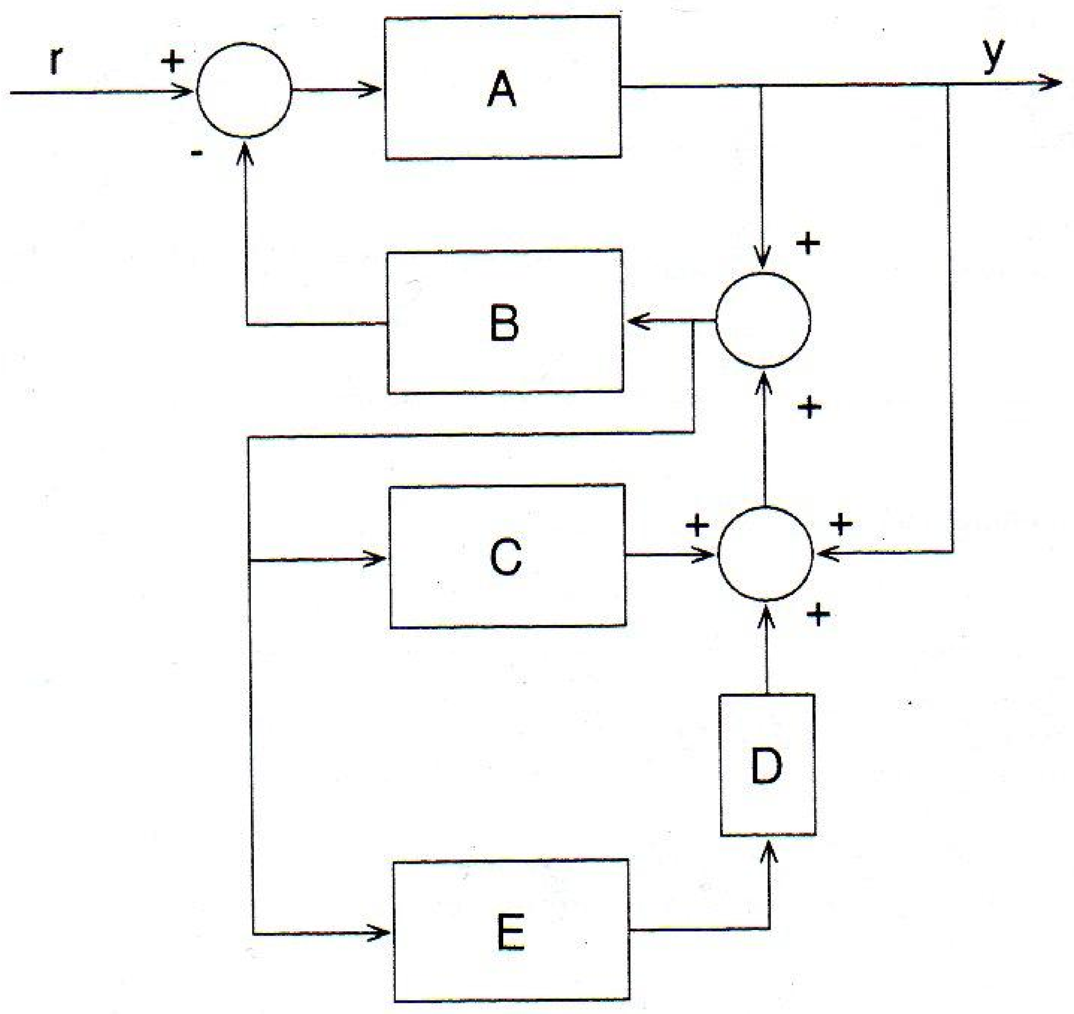


Figure 2: Block diagram for Problem 2.

1. Compute the transfer function from R to Y.
2. Assume now that the transfer function from R to Y is given by:

$$H(s) = \frac{s^2 + s + 2}{s^3 + s^2 + ks + 3k - 5}$$

   (a) for which values of k is the system stable?

   (b) what is the steady state error to a unitary step input when $k = 2$?

   (c) design a controller to force the steady state error to become zero when $k = 2$.

Problem 1:

1. $$\frac{dx_2}{dt} = -\sigma(x_2 - x_1)$$

2. $$\begin{cases} \dfrac{dx_2}{dt} = -\sigma(x_2 - x_1) \\ \dfrac{dx_1}{dt} = -\alpha x_1 - \beta(x_1 - x_2) + u \end{cases} \quad \overset{\mathcal{L}\text{ transform}}{\Longleftrightarrow}$$

$$\begin{cases} S X_2(s) = -\sigma(X_2(s) - X_1(s)) \Rightarrow X_1(s) = \dfrac{s+\sigma}{\sigma} X_2(s) \\ S X_1(s) = -\alpha X_1(s) - \beta(X_1(s) - X_2(s)) + U(s) \quad (1) \end{cases}$$

with (1) $\Rightarrow$ $$(s+\alpha+\beta) X_1(s) = \beta X_2(s) + U(s)$$

$$(s+\alpha+\beta)\frac{s+\sigma}{\sigma} X_2(s) = \beta X_2(s) + U(s)$$

$$\frac{X_2(s)}{U(s)} = \frac{1}{(s+\alpha+\beta)(1+\frac{s}{\sigma}) - \beta}$$

3. (a) Laplacian transform:

$$\begin{cases} S X_1(s) = -\alpha X_1(s) - X_2(s) + U(s) \quad (2) \\ S X_2(s) = \beta X_1(s) - \alpha X_2(s) \Rightarrow X_1(s) = \dfrac{s+\alpha}{\beta} X_2(s) \end{cases}$$

with (2) $$(s+\alpha) X_1(s) = -X_2(s) + U(s)$$

$$\frac{(s+\alpha)^2}{\beta} X_2(s) = -X_2(s) + U(s)$$

$$\left(1 + \frac{(s+\alpha)^2}{\beta}\right) X_2(s) = U(s)$$

$$\frac{X_2(s)}{U(s)} = \frac{\beta}{\beta + (s+\alpha)^2} = \frac{\beta}{s^2 + 2\alpha s + \alpha^2 + \beta}$$

Problem 1:

(b) This is a second order system

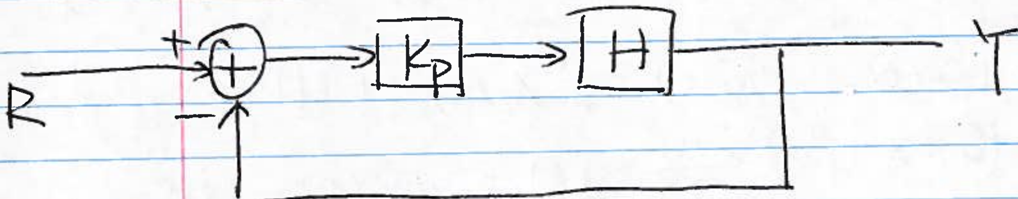
i. $t_r \simeq \frac{1.8}{w_n} \leq 0.45 \quad w_n \geq 4$

ii. $t_s = \frac{4.6}{\sigma} \leq 9.2 \quad \sigma \geq \frac{1}{2}$

$$w_n^2 = 16 = \alpha^2 + \beta \geq 16 \qquad \begin{cases} \alpha \geq \frac{1}{2} \\ \alpha^2 + \beta \geq 16 \end{cases}$$

$$\sigma = \alpha \geq \frac{1}{2}$$

(c) $H(s) = \dfrac{\beta}{s^2 + 2\alpha s + \alpha^2 + \beta} = \dfrac{1}{s^2 + 2s + 2}$



$$\frac{Y}{R} = \frac{K_p H}{1 + K_p H} = \frac{K_p}{s^2 + 2s + 2 + K_p}$$

$$\begin{cases} w_n \geq 4 \\ \sigma \geq \frac{1}{2} \end{cases} \Rightarrow \begin{cases} 2 + K_p \geq 16 \\ 1 \geq \frac{1}{2} \end{cases} \Rightarrow K_p \geq 14$$

(d) From the question, we know $\frac{dx_2}{dt} = \beta x_1 - \alpha x_2$

$$s X_2(s) = \beta X_1(s) - \alpha X_2(s)$$

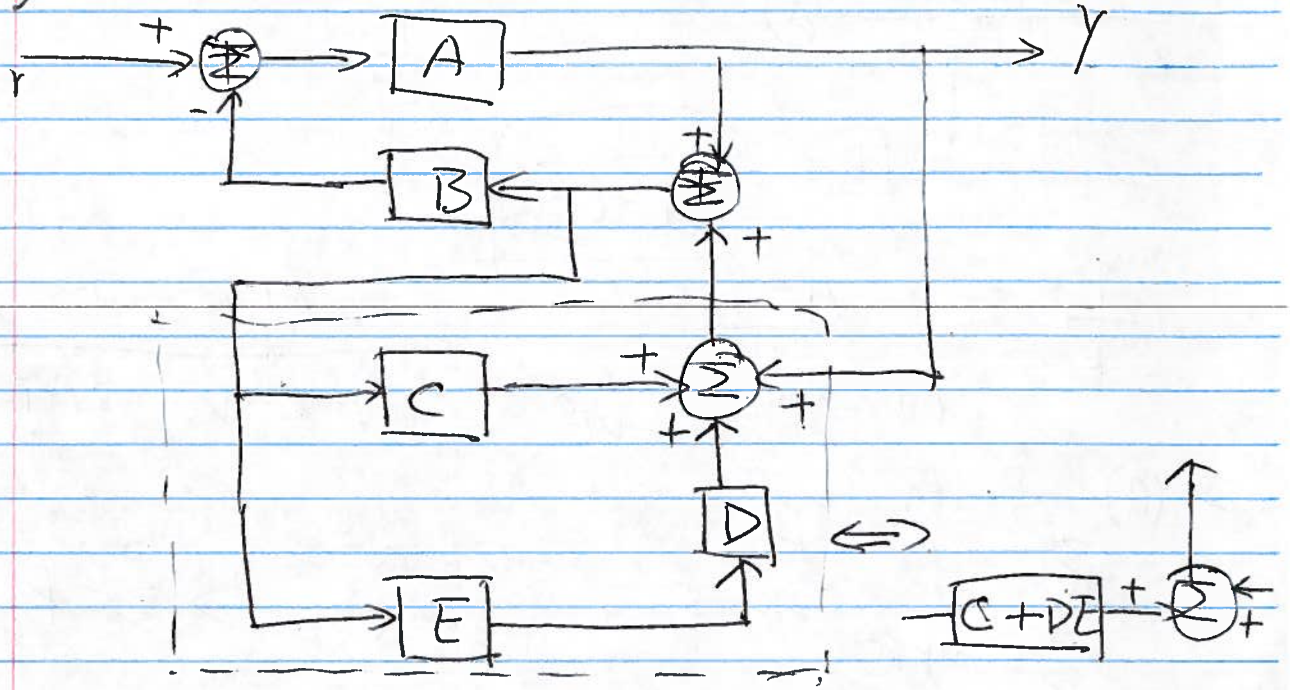
$$\beta X_1(s) = (s + \alpha) X_2(s) \qquad X_2(s) = \frac{\beta}{s+\alpha} X_1(s)$$

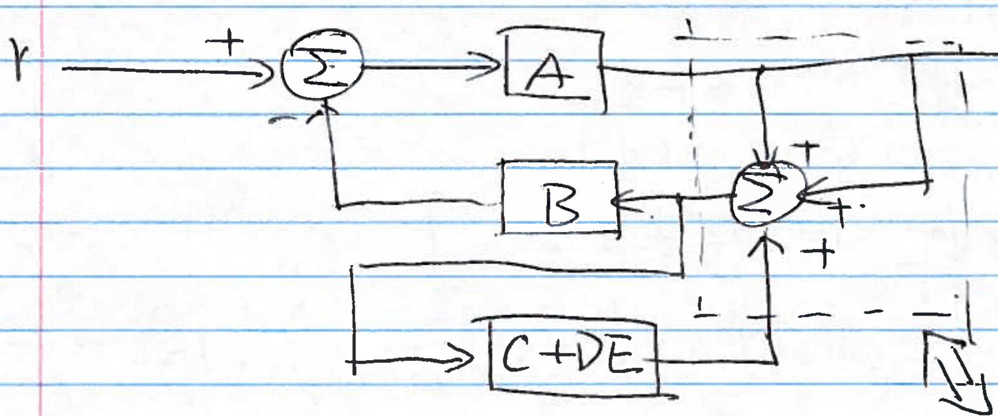
As $\lim_{s \to 0} s X_2(s) = 20$

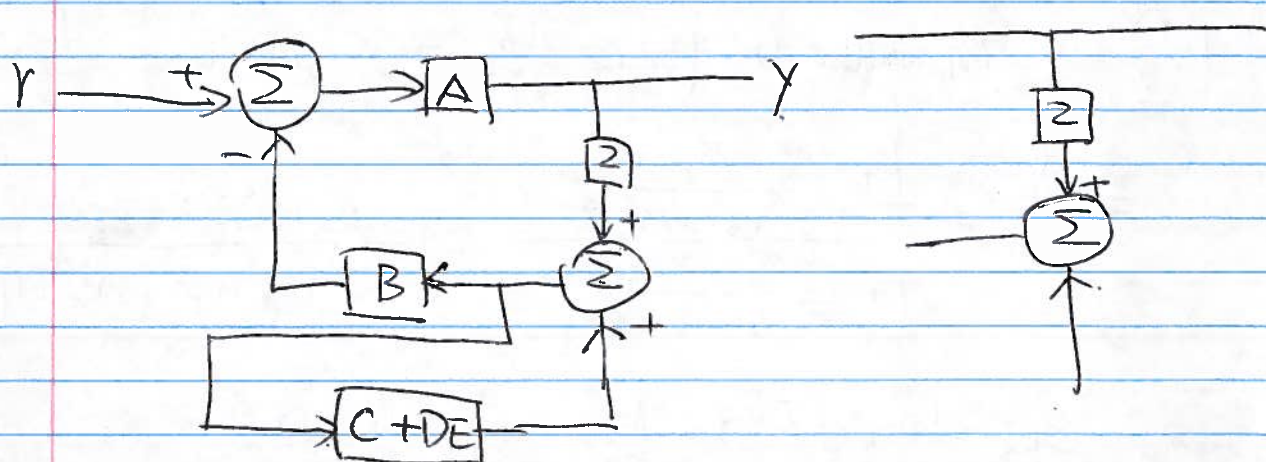
$$\lim_{s \to 0} s X_1(s) = \lim_{s \to 0} s \cdot \frac{s+\alpha}{\beta} X_2(s) = \lim_{s \to 0} 20 \frac{\alpha}{\beta} = 20$$

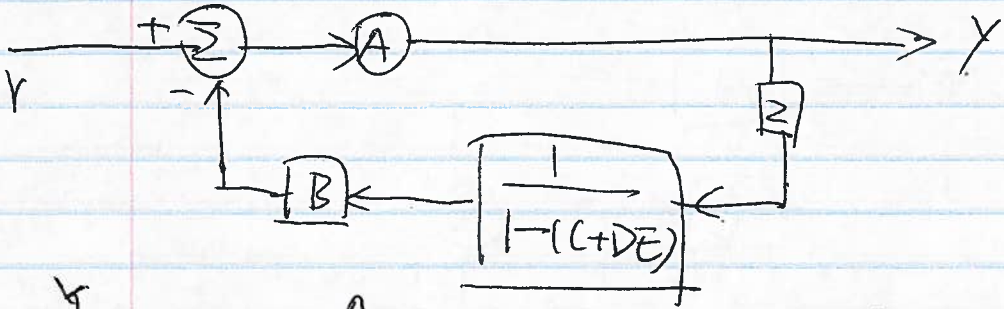
Problem 3:

1.









$$\frac{Y}{R} = \frac{A}{1 + 2AB\,\frac{1}{1-C-DE}} = \frac{A(1-C-DE)}{1-C-DE+2AB}$$

2.(a) Routh test

$s^3: \quad 1 \quad K$

$s^2: \quad 1 \quad 3K-5$

$s^1: \quad -2K+5$

$s^0: \quad 3K-5$

$$\begin{cases} -2K+5>0 \\ 3K-5>0 \end{cases} \Rightarrow \frac{5}{3} < K < \frac{5}{2}$$

(b) $H(s) = \dfrac{s^2+s+2}{s^3+s^2+2s+1}$

$e_{ss} = \cancel{\lim_{s\to 0} s\left(H(s) - \frac{1}{s}\right) = -1}$

$$\lim_{s\to 0} s\left(\frac{1}{s} - \frac{1}{s}\,\frac{s^2+s+2}{s^3+s^2+2s+1}\right) = 1-2 = -1$$

(c) In order to force $e_{ss}=0$, we use $\frac{k_I}{s}$ controller

$$\Rightarrow \frac{Y}{R} = \frac{\frac{k_I}{s}\,\frac{s^2+s+2}{s^3+s^2+2s+1}}{1+\frac{k_I}{s}\,\frac{s^2+s+2}{s^3+s^2+2s+1}} = \frac{k_I(s^2+s+2)}{s^4+s^3+(k_I+2)s^2+(k_I+1)s+2k_I}$$

$$e_{ss} = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s\left(1-\frac{Y}{R}\right)R$$

$$= \lim_{s \to 0} s\left(1 - \frac{K_2(s^2+s+2)}{s^4+s^3+(K_2+2)s^2+(K_2+1)s+2K_2}\right)\cdot\frac{1}{s}$$

$$= 0$$

Check the stability of the system with Routh test

$$\begin{array}{llll}
s^4 : & 1 & K_2+2 & 2K_2 \\
s^3 : & 1 & K_2+1 & \\
s^2 : & 1 & 2K_2 & \\
s^1 : & 1-K_2 & & \\
s^0 : & 2K_2 & &
\end{array}$$

$$\begin{cases} 1-K_2 > 0 \\ 2K_2 > 0 \end{cases} \Rightarrow 0 < K_2 < 1$$