# ECE 141 – Principles of Feedback Control
**Midterm Solutions**

**Problem** (1 - Modeling and Design of Motorized Propeller)**.**

**Solution. (a) + (b) + (c)** Collecting all of the implied dynamics in an ODE, we have the complete electromechanical model of the (zero-resistance/reactive) electrical motorized propeller. $\theta$ is the angle of the propeller, $i$ is the electric current through the motor, and $V$ is the applied voltage to the motor.

$$\mu\frac{d^2\theta}{dt^2} = -k\frac{d\theta}{dt} + \beta i(t) \qquad L\frac{di}{dt} = V(t) - \psi\frac{d\theta}{dt} \qquad \tau = \gamma\frac{d\theta}{dt}$$

$\mu$ is the rotational moment of inertia of the rotors. $k$ is the frictional constant of the motor. $\beta$ is some linearized amalgamation of coil area, and magnetic field. $L$ is the motor coil inductance. $\psi$ is some linearized approximation of oscillating magnetic flux. $\gamma$ is some linearized amalgamation of the rotor blade cross-sectional area and air density/composition that converts angular velocity to momentum/thrust $\tau$.

Assuming zero initial conditions and applying the Laplace Transform, it follows that:

$$\mu s^2 \cdot \theta(s) = -ks \cdot \theta(s) + \beta \cdot I(s) \qquad Ls \cdot I(s) = V(s) - \psi s \cdot \theta(s) \qquad \tau(s) = \gamma s \cdot \theta(s)$$

Solving for the applied voltage to output thrust transfer function $G(s) = \dfrac{\tau(s)}{V(s)}$, we have:

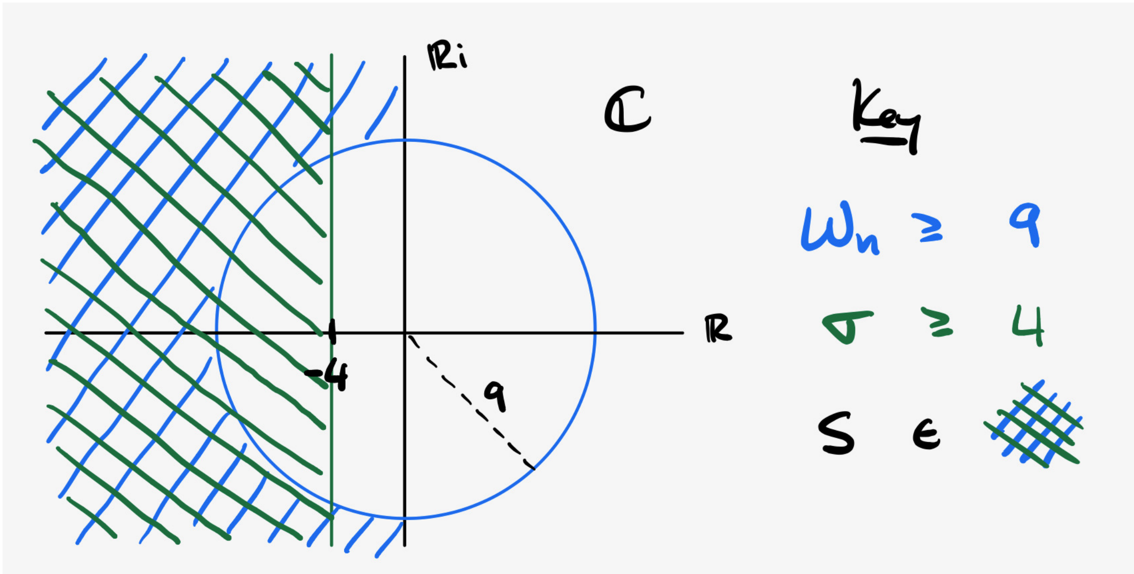
$$G(s) = \frac{\tau(s)}{V(s)} = \frac{\gamma\beta}{\mu L s^2 + kLs + \psi\beta} = \frac{\gamma}{\psi} \cdot \frac{\frac{\psi\beta}{\mu L}}{s^2 + \frac{k}{\mu}s + \frac{\psi\beta}{\mu L}}$$

**(d) + (e)** In particular, assume that the parameters of the motorized propeller system set the poles of the transfer function $G(s)$ to $s_1 = -8 + 4i$ and $s_2 = -8 - 4i$ with $G(0) = 1/80$.

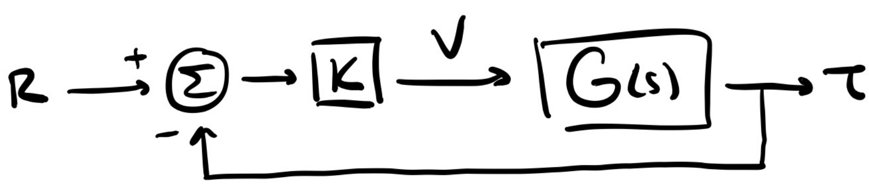
$$G(s) = \frac{1}{s^2 + 16s + 80}$$

To have the rise time $t_r = 1.8/\omega_n \leq 0.2$ seconds and the settling time $t_s = 4.6/\sigma = 4.6/\zeta\omega_n \leq 1.15$ seconds, we require that:

$$t_r = \frac{9/5}{\omega_n} \leq \frac{1}{5} \implies \omega_n \geq 9$$
$$t_s = \frac{23/5}{\sigma} \leq \frac{23}{20} \implies \sigma \geq 4$$




To design a controller for $G(s)$ that satisfies the design constraints and tracks a step reference input with steady state thrust error less than or equal to 16/17, we construct a proportional negative feedback control system with proportional constant $K$.




$$H(s) = \frac{\tau}{R} = \frac{KG(s)}{1 + KG(s)} = \frac{K}{s^2 + 16s + 80 + K}$$

Comparing coefficients of the second-order closed-loop transfer function $H(s)$, we observe that $\sigma = 8 \geq 4$ is independent of $K$, and $\omega_n = \sqrt{80 + K} \geq 9$ necessitates $K \geq 1$. Note that:

$$sE(s) = s\left[R(s) - \tau(s)\right] = s\left[1 - H(s)\right]R(s) = 1 - \frac{K}{s^2 + 16s + 80 + K} = \frac{s^2 + 16s + 80}{s^2 + 16s + 80 + K}$$

$$D(s) = s^2 + 16s + 80 + K = 0 \implies s = \frac{-16 \pm \sqrt{256 - 4(80 + K)}}{2} \implies \text{Real}(s) < 0 \text{ for all } K \geq 1$$

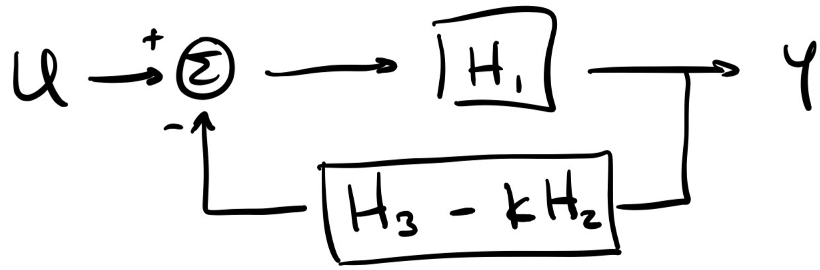
Thus, we can utilize the final value property of the Laplace Transform. To set the steady state thrust error of $H(s)$ in response to a unit step reference input $R(s) = s^{-1}$ below 16/17, we require that:

$$\lim_{t \to \infty} \left[e(t) = r(t) - \tau(t)\right] = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\left[1 - H(s)\right]R(s) = \frac{1}{1 + KG(0)} = \frac{80}{80 + K} \leq \frac{16}{17} \implies K \geq 5$$

Hence, the proportional negative feedback controller with $K \geq 5$ applied to the motorized propeller system $G(s)$ tracks unit step inputs with error below the acceptable bound of 16/17 and satisfies the design constraints $t_r \leq 0.2$ seconds and $t_s \leq 1.15$ seconds.

**Problem** (2 - Block Diagrams and Steady State)**.**

**Solution.** **(a)** Re-arranging the blocks, observe that:




$$H(s) = \frac{Y(s)}{U(s)} = \frac{H_1}{1 + (H_3 - kH_2)H_1} = \frac{-1}{1 - [s^3 + s^2 - k\,(s+4)]} = \frac{1}{s^3 + s^2 - ks - 4k - 1}$$

Construct the Routh diagram to analyze the stability of the poles of $H(s)$.

$$\begin{array}{c|cc} s^3 & 1 & -k \\ s^2 & 1 & -(1+4k) \\ s^1 & 1+3k & 0 \\ s^0 & -(1+4k) & \end{array}$$

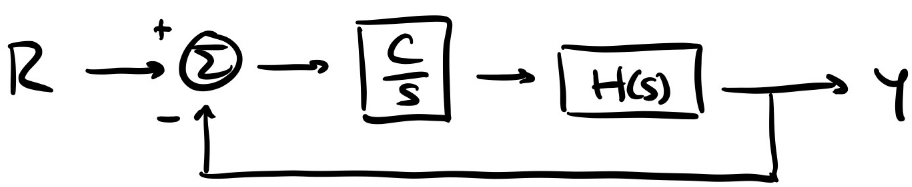
Necessarily, taking $k \in (-1/3, -1/4)$ sets all poles to satisfy $\text{Real}(s) < 0$ such that $H(s)$ is stable.

**(b)** Assume $k = -3/10$. Note that $k$ induces stable poles for $H(s)$ and $sE(s) = H(s) - 1$. Observe that:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,[Y(s) - U(s)] = \lim_{s\to 0} H(s) - 1 = \lim_{s\to 0} \frac{1}{s^3 + s^2 + 3s/10 + 6/5 - 1} - 1 = 4 \neq 0$$

Hence, the control system cannot track unit step inputs $U(s) = s^{-1}$.

**(c)** Assume $k = -3/10$. To eliminate constant steady state error, we utilize an integral controller.




$$T(s) = \frac{Y(s)}{R(s)} = \frac{Cs^{-1}H(s)}{1 + Cs^{-1}H(s)} = \frac{C}{s^4 + s^3 - ks^2 - (1+4k)s + C}$$

Necessarily, the control system $T(s)$ must be stable. Applying the Routh test, we design $C$.

$$\begin{array}{c|ccc} s^4 & 1 & -k & C \\ s^3 & 1 & -(1+4k) & 0 \\ s^2 & 1+3k & C & \\ s^1 & -\frac{C+(1+3k)(1+4k)}{1+3k} & 0 & \\ s^0 & C & & \end{array} \implies \left\{ \begin{array}{c} C > 0 \\ C < -(1+3k)(1+4k) = 1/50 \end{array} \right\} \implies C \in (0, 0.02)$$

Analyzing the steady-state error in response to a unit step $R(s) = s^{-1}$, we have:

$$\lim_{s\to 0} s\,[Y(s) - R(s)] = \lim_{s\to 0} s\,[T(s) - 1]\,R(s) = \lim_{s\to 0} T(s) - 1 = \lim_{s\to 0} -\frac{s^4 + s^3 - ks^2 - (1+4k)s}{s^4 + s^3 - ks^2 - (1+4k)s + C} = 0$$