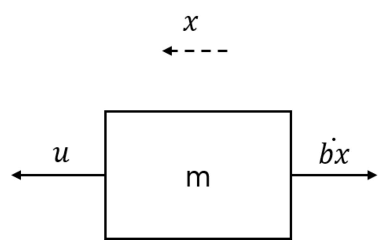
Prob 1.



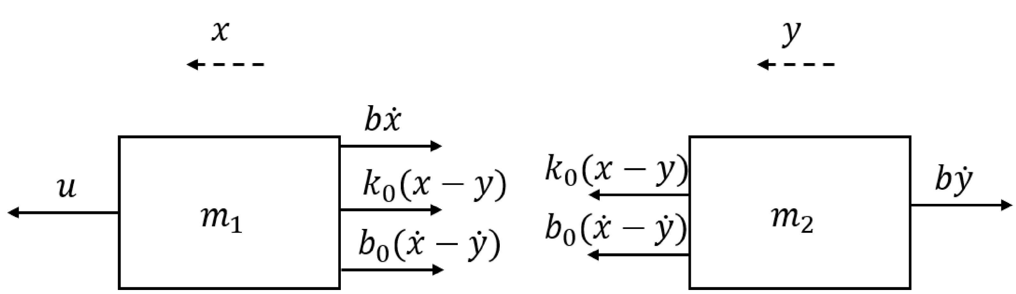
We denote by x the position of the car, by $\dot{x}$ its velocity, and by $\ddot{x}$ its acceleration. Hence, by Newton's law of motion we have:

$$m\ddot{x} = -b\dot{x} + f,$$

where f is the force exerted by the car's engine.

Prob 2.

a)



We now use x for the position of car 1 and y for the position of car 2. Using Newton's law again we obtain:

$$m\ddot{x} = u - b\dot{x} - k(x-y) - d(\dot{x} - \dot{y}) \tag{1}$$
$$m\ddot{y} = -b\dot{y} - k(y-x) - d(\dot{y} - \dot{x}) \tag{2}$$

b) Since we have two cars, we reproduce the equations derived in Problem 1 twice (once for each car):

$$m\ddot{x} = f_1 - b\dot{x} \tag{3}$$
$$m\ddot{y} = f_2 - b\dot{y}. \tag{4}$$

In order for the two cars described by equations (3) and (4) to behave like the system in Figure 1 in the exam, we need to design the inputs f_1 and f_2 so that equations (3) and (4) become equal to the equations (1) and (2). This leads to:

$$f_1 = u - k(x-y) - d(\dot{x} - \dot{y}) \tag{5}$$
$$f_2 = -k(y-x) - d(\dot{y} - \dot{x}). \tag{6}$$

Prob 3.
If the controllers (5) and (6) are being used, the equations of motion describing the evolution of the cars' positions are (1) and (2). Subtracting (2) from (1) we obtain:

$$m(\ddot{x} - \ddot{y}) = u - b(\dot{x} - \dot{y}) - 2k(x-y) - 2d(\dot{x} - \dot{y}) \tag{7}$$

If we replace $x - y$ with z, equation (7) becomes:

$$m\ddot{z} = u - b\dot{z} - 2kz - 2d\dot{z} \tag{8}$$
$$u = m\ddot{z} + (b+2d)\dot{z} + 2kz. \tag{9}$$

Applying the Laplace transform on both sides (assuming zero initial conditions) we obtain:

$$U = (ms^2 + (b+2d)s + 2k)Z.$$

Therefore, the transfer function from u to z is

$$\frac{Z}{U} = \frac{1/m}{s^2 + \frac{b+2d}{m}s + \frac{2k}{m}}.$$

Prob 4. The output Z to a ramp input $U = \frac{1}{s^2}$ is given by:

$$Z(s) = \frac{1/m}{s^2 + \frac{b+2d}{m}s + \frac{2k}{m}} \frac{1}{s^2}. \tag{10}$$

We can perform a partial fraction expansion to obtain:

$$Z(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \frac{c_3}{s} + \frac{c_4}{s^2}, \tag{11}$$

where c_1, c_2, c_3 and c_4 are constants and p_1 and p_2 are the roots of $s^2 + \frac{b+2d}{m}s + \frac{2k}{m}$. Applying the inverse Laplace transform we obtain:

$$z(t) = c_1 e^{p_1 t}u(t) + c_2 e^{p_2 t}u(t) + c_3 u(t) + c_4 t u(t).$$

In order to have $\lim_{t\to\infty} z(t) = 0$ we need $\text{Re}(p_1) < 0$, $\text{Re}(p_2) < 0$, $c_3 = 0$, and $c_4 = 0$. The constant c_4 is given by:

$$c_4 = s^2 Z(s)\Big|_{s=0} = \left.\frac{1/m}{s^2 + \frac{b+2d}{m}s + \frac{2k}{m}}\right|_{s=0} = 1/2k,$$

and we see that we cannot make c_4 to be zero by choosing the value of the constant k.

Prob 5. Let's first try a proportional controller with gain K_p. The closed-loop transfer function is:

$$\frac{\frac{1/m}{s^2+\frac{b+2d}{m}s+\frac{2k}{m}}K_p}{1+\frac{1/m}{s^2+\frac{b+2d}{m}s+\frac{2k}{m}}K_p} = \frac{\frac{1}{m}K_p}{s^2 + \frac{b+2d}{m} + \frac{2k+K_p}{m}}. \tag{12}$$

For a ramp input $U = \frac{1}{s^2}$ we obtain:

$$Z = \frac{\frac{1}{m}K_p}{s^2 + \frac{b+2d}{m} + \frac{2k+K_p}{m}} \frac{1}{s^2}. \tag{13}$$

Performing again a partial fraction expansion:

$$Z(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \frac{c_3}{s} + \frac{c_4}{s^2}, \tag{14}$$

where p_1 and p_2 are the roots of $s^2 + \frac{b+2d}{m}s + \frac{2k+k_p}{m}$, we observe that c_4 must be zero for $\lim_{t\to\infty} z(t)$ to be bounded. Computing c_4 as in the previous question we obtain:

$$c_4 = \frac{K_p}{2k+K_p},$$

which is only zero if $K_p = 0$ which is not possible: if the controller is placed on the feedback path this corresponds to having no controller; if the controller is placed on the feedforward path this corresponds to disconnecting the input from the plant.

Hence, we now seek a PD controller. The output produced by a ramp is:

$$Z = \frac{\frac{1/m}{s^2+\frac{b+2d}{m}s+\frac{2k}{m}}(K_d s + K_p)}{1 + \frac{1/m}{s^2+\frac{b+2d}{m}s+\frac{2k}{m}}(K_d s + K_p)} \frac{1}{s^2} \tag{15}$$

$$= \frac{\frac{1}{m}(K_d s + K_p)}{s^2 + \frac{b+2d+K_d}{m}s + \frac{2k+K_p}{m}} \frac{1}{s^2} \tag{16}$$

$$= \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \frac{c_3}{s} + \frac{c_4}{s^2}. \tag{17}$$

In order for $\lim_{t\to\infty} z(t)$ to be bounded we need: 1) $c_4 = 0$ and 2) $\mathrm{Re}(p_1) < 0$ and $\mathrm{Re}(p_2) < 0$.

The constant c_4 is given by:

$$c_4 = s^2 Z(s)\big|_{s=0} = \left.\frac{(K_d s + K_p)/m}{s^2 + \frac{b+2d+K_d}{m}s + \frac{2k+K_p}{m}}\right|_{s=0} = \frac{K_p}{2k+K_p},$$

and thus we take $K_p = 0$.

To enforce $\mathrm{Re}(p_1) < 0$ and $\mathrm{Re}(p_2) < 0$ we do a Routh test that provides:

$$\frac{b+2d+K_d}{m} > 0, \qquad \frac{2k}{m} > 0.$$

Since k and m are positive, $2k/m$ is always positive. Hence, we only have the constraint:

$$K_d > -b - 2d.$$

If K_d satisfies the above constraint and $K_p = 0$, the inverse Laplace transform provides:

$$\mathcal{L}^{-1}\{Z\} = \mathcal{L}^{-1}\left\{\frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \frac{c_3}{s}\right\} = c_1 e^{-p_1 t}u(t) + c_2 e^{-p_2 t}u(t) + c_3 u(t),$$

and $\lim_{t\to\infty} z(t) = c_3$. We can compute the constant c_3 as:

$$sZ(s)|_{s=0} = \left.\frac{\frac{K_d s}{m}}{s^2 + \frac{b+2d+K_d}{m}s + \frac{2k}{m}} \frac{1}{s}\right|_{s=0} = \frac{K_d}{2k}. \tag{18}$$

Therefore, we can achieve $\lim_{t\to\infty} |z(t)| \leq 0.1$ with $K_d \leq 0.2k$. Summarizing the design, we have $K_p = 0$ and K_d can be any constant in the set $[-b - 2d, 0.2k]$.

Prob 6.

We start with a proportional controller resulting in the closed-loop transfer function:

$$\frac{Z}{R} = \frac{\frac{1}{m}K_p}{s^2 + \frac{b+2d}{m}s + \frac{2k+K_p}{m}}. \tag{19}$$

The error is given by:

$$E(s) = R - Z = \left(1 - \frac{\frac{1}{m}K_p}{s^2 + \frac{b+2d}{m}s + \frac{2k+K_p}{m}}\right) R \tag{20}$$

$$= \frac{s^2 + \frac{b+2d}{m}s + \frac{2k}{m}}{s^2 + \frac{b+2d}{m}s + \frac{2k+K_p}{m}} R. \tag{21}$$

To make the system stable we first do a Routh test that leads to:

$$\frac{b+2d}{m} > 0, \qquad \frac{2k+K_p}{m} > 0.$$

Since all the constans are positive, this lead to the constraint:

$$K_p > -2k.$$

Assuming that we pick K_p satisfying this constraint, the system is stable and we can use the Final Value Theorem to compute the steady state error:

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{s^2 + \frac{b+2d}{m}s + \frac{2k}{m}}{s^2 + \frac{b+2d}{m}s + \frac{2k+K_p}{m}}\frac{1}{s} \tag{22}$$

$$= \frac{2k}{2k+K_p} \tag{23}$$

$$= \frac{400}{400+K_p} \tag{24}$$

Since we want $|e_{ss}| = \left|\frac{400}{400+k_P}\right| \leq 0.01$, we need a gain K_p satisfying:

$$K_p \geq 39600. \tag{25}$$

Since we have a transfer function with 2 zeros and no poles, we can relate the settling time t_s to ω_n and ζ by:

$$t_s = \frac{4.6}{\omega_n\zeta} < 10.$$

We note that:

$$2\omega_n\zeta = \frac{b+2d}{m} = 1.12 \implies \frac{4.6}{\omega_n\zeta} = \frac{4.6}{0.56} < 10, \tag{26}$$

and the settling time specification is already satisfied.
For the rise time we have $t_r = 1.8/\omega_n \leq 1$ leading to:

$$\omega_n \geq 1.8 \tag{27}$$

$$\sqrt{\frac{2k+K_p}{m}} \geq 1.8 \tag{28}$$

$$\frac{2k+K_p}{m} \geq 1.8^2 \tag{29}$$

$$2k + K_p \geq m1.8^2 \tag{30}$$

$$K_p \geq m1.8^2 - 2k \tag{31}$$

We know that $2 \geq 1.8$ implies $4 = 2^2 \geq 1.8^2$, hence it suffices to find K_p satisfying:

$$K_p \geq 10000 \cdot 4 - 400 = 3600.$$

Summarizing, we need to choose K_p satisfying:

$$K_p \geq \max\{39600, 3600\} = 39600.$$