EE 141 – Midterm
Fall 2009

04/11/09

Duration: 1 hour and 40 minutes

The midterm is closed book and closed lecture notes. No calculators.
You can use a single page of handwritten notes.
Please carefully justify all your answers.

Problem 1: Consider the truck represented in Figure 1. We model the connection between the trailer and the truck by a spring with restitution coefficient K and a damper with damping coefficient b. The truck has mass m_1 and the trailer has mass m_2. The truck engine applies a force f as represented in Figure 1. In addition to the force f, there is also a friction force acting on the truck and a friction force action on the trailer. The friction forces are proportional to the velocity with proportionality constant c.

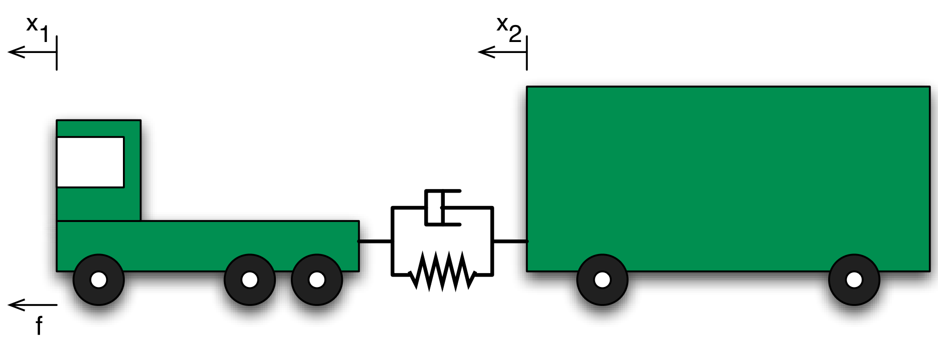


Figure 1: Truck model for Problem 1.

1. What are the differential equations describing the evolution of the truck position x_1 and the trailer position x_2?
2. What is the transfer function from f to x_2?
3. Knowing that $K = 13000\ N/m$, $b = 1100\ Ns/m$, $c = 160\ Ns/m$, $m_1 = 2000\ Kg$, and $3000\ Kg$, what is $\lim_{t\to\infty} \frac{d}{dt} x_2$ when you apply a step input of magnitude 5?

Problem 2: Consider the block diagram in Figure 2.

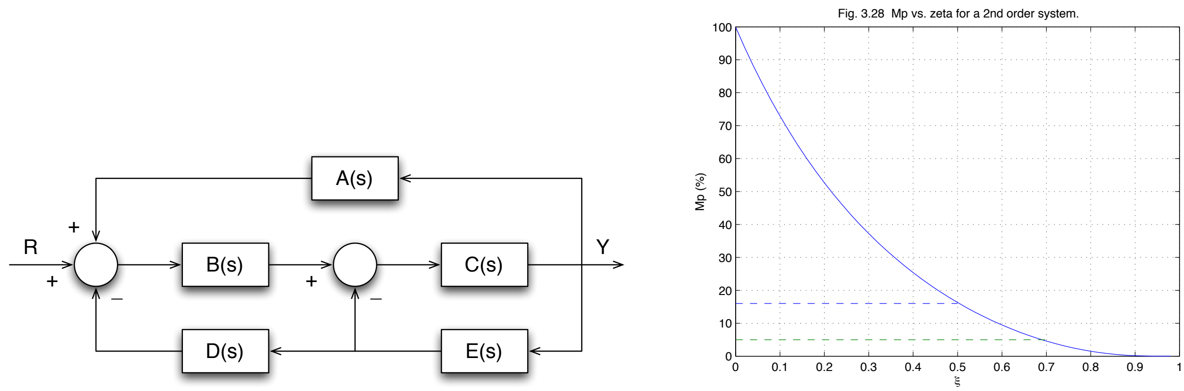


Figure 2: Left: block diagram for Problem 2. Right: overshoot vs damping ration for a second order system with no zeros.

1. Compute the transfer function from from R to Y.

2. Assume now that the transfer function from R to Y is given by:

$$\frac{Y}{R} = \frac{b_0}{a_2 a^2 + a_1 s + a_0},$$

   and that you can design the parameters b_0, a_2, a_1, and a_0. Which values would you choose for these parameters so that the following specifications are met:

   (a) $\lim_{t\to\infty} y(t) = 2$ when the input is a unit step;

   (b) rise time smaller than 1.8 seconds;

   (c) overshoot smaller than 5%;

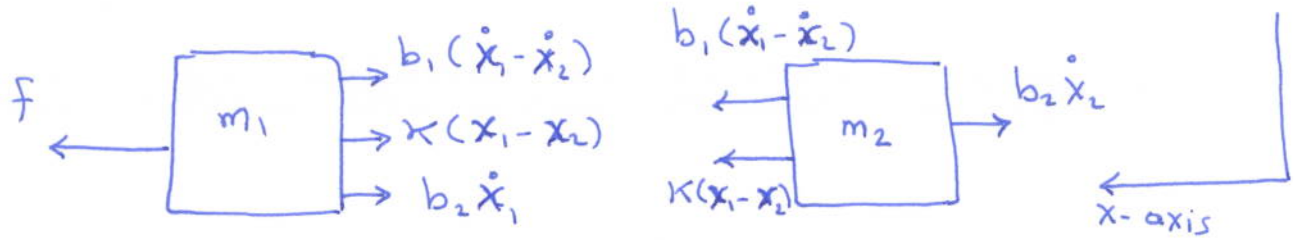
   (d) settling time smaller than 10 seconds.

Problem 3: Consider the following transfer function:

$$H(s) = \frac{s^2 + 12s + 10}{s^3 + 4s^2 + 14s + 20}$$

1. Is the system described by $H(s)$ stable?
2. Can you track step inputs with a proportional controller in a unit feedback loop?
3. Design a controller to be used in a unit feedback loop so that the closed-loop system can track step inputs.

1.1]



Truck:

$$f - b_1(\dot{x}_1 - \dot{x}_2) - K(x_1 - x_2) - b_2\dot{x}_1 = m_1\ddot{x}_1 \quad \dots\dots (1)$$

Trailer:

$$b_1(\dot{x}_1 - \dot{x}_2) + K(x_1 - x_2) - b_2\dot{x}_2 = m_2\ddot{x}_2 \quad \dots (2)$$

(1): $$f(t) + 1100\,\dot{x}_2(t) + 13000\,x_2(t) = 2000\,\ddot{x}_1(t) + 160\,\dot{x}_1(t) + 1100\,\dot{x}_1(t) + 13000\,x_1(t)$$

(2) $$1100\,\dot{x}_1(t) + 13000\,x_1(t) = 3000\,\ddot{x}_2(t) + 1260\,\dot{x}_2(t) + 13000\,x_2(t)$$

1.2] (1) $$F(s) + 1100\,sX_2(s) + 13000\,X_2(s) = 2000\,s^2X_1(s) + 1260\,sX_1(s) + 13000\,X_1(s)$$

$$F(s) + X_2(s)\,[1100s + 13000] = X_1(s)\,[2000s^2 + 1260s + 13000]$$

(2) $$1100\,sX_1(s) + 13000\,X_1(s) = 3000\,s^2X_2(s) + 1260\,sX_2(s) + 13000\,X_2(s)$$

$$X_1(s)\,[1100s + 13000] = X_2(s)\,[3000s^2 + 1260s + 13000]$$

From 1 and 2:

$$F(s) + X_2(s)\,[1100s + 13000] = X_2(s)\,[3000s^2 + 1260s + 13000] \times \frac{[2000s^2 + 1260s + 13000]}{1100s + 13000}$$

$$F(s) + X_2(s)\,[sb_1 + K] = X_2(s)\,\frac{[m_1s^2 + (b_1+b_2)s + K][m_2s^2 + (b_1+b_2)s + K]}{sb_1 + K}$$

$$F(s) = X_2(s)\left[\frac{m_1m_2s^4 + s^2(m_1+m_2)[s(b_1+b_2)+K] + [s(b_1+b_2)+K]^2 - [sb_1+K]^2}{sb_1+K}\right]$$

$$= X_2(s)\left[\frac{m_1m_2s^4 + s^2(m_1+m_2)[s(b_1+b_2)+K] + 2sb_2(sb_1+K) + s^2b_2^2}{sb_1+K}\right]$$

$$= X_2(s)\left[\frac{m_1m_2s^4 + s^3[m_1+m_2][b_1+b_2] + s^2[K(m_1+m_2) + b_2^2 + 2b_1b_2] + 2b_2Ks}{sb_1+K}\right]$$

$$\therefore \frac{X_2(s)}{F(s)} = \frac{1100s + 13{,}000}{6{,}000{,}000s^4 + 6{,}300{,}000s^3 + 65{,}377{,}600s^2 + 4{,}160{,}000s}$$

1.3] $$\lim_{t\to\infty} \dot{x}_2(t) = \lim_{s\to 0}[s.sX_2(s)] = \lim_{s\to 0} s^2.[\quad].F(s) = \lim_{s\to 0} s^2[\quad].\frac{8000}{s}$$

$$= \frac{13{,}000 \times 8000}{4160000} = 25 \text{ m/s}$$

Problem 2

(1) $Y = [(R - DEY + AY)B - EY]C$

$$\Rightarrow Y = \frac{BC}{1 + BCDE - ABC + CE} R$$

$$\frac{Y(s)}{R(s)} = \frac{B(s)\,C(s)}{1 + B(s)C(s)D(s)E(s) - A(s)B(s)C(s) + C(s)E(s)}$$

(2) $$\frac{Y(s)}{R(s)} = \frac{b_0}{a_2 s^2 + a_1 s + a_0} = \frac{\frac{b_0}{a_2}}{s^2 + \frac{a_1}{a_2} s + \frac{a_0}{a_2}} = \frac{\frac{b_0}{a_2}}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

Specification (a) $\Rightarrow$ $y(\infty) = \lim_{s \to 0} s \frac{b_0}{a_2 s^2 + a_1 s + a_0} \frac{1}{s} = \frac{b_0}{a_0} = 2$

specification (b) $\Rightarrow$ $t_r = \frac{1.8}{\omega_n} < 1.8$

$\Rightarrow \omega_n > 1$

specification (c) $\Rightarrow$ $M_p < 0.05$ $\Rightarrow$ $\zeta > 0.7$

specification (d) $\Rightarrow$ $t_s = \frac{4.6}{\zeta \omega_n} < 10$ $\Rightarrow$ $\zeta \omega_n > 0.46$

(d) is satisfied if (b) (c) are satisfied

Pick $\zeta = 0.7$, $\omega_n = 1$

(a) $\Rightarrow$ $\frac{b_0}{a_0} = 2$

(b) $\Rightarrow$ $\frac{a_0}{a_2} = \omega_n^2 = 1$

(b) (c) $\Rightarrow$ $\frac{a_1}{a_2} = 2\zeta\omega_n = 1.4$

Then we can select $a_2 = 1$, then $a_0 = 1$. $a_1 = 1.4$, $b_0 = 2$

3] 1. $G(s) = \dfrac{s^2 + 12s + 10}{s^3 + 4s^2 + 14s + 20}$

using Routh's criterion:

$$\begin{array}{lll} s^3 & 1 & 14 \\ s^2 & 4 & 20 \\ s^1 & 9 & \\ s^0 & 20 & \end{array}$$

- All elements of the 1st column and all of the coefficients of the characteristic polynomial are > 0

$\therefore$ The system is stable.

alternatively: solving $s^3 + 4s^2 + 14s + 20 = 0$

-2 is a root:

$$\begin{array}{r} s^2 + 2s + 10 \\ s+2 \,\overline{)\, s^3 + 4s^2 + 14s + 20} \\ \underline{s^3 + 2s^2 } \\ 2s^2 + 14s + 20 \\ \underline{2s^2 + 4s } \\ 10s + 20 \\ \underline{10s + 20} \\ 0 \end{array}$$

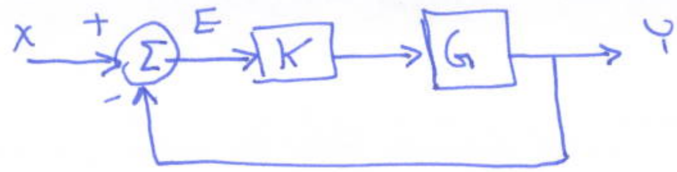
$\therefore (s+2)(s^2 + 2s + 10) = 0$

solving $s^2 + 2s + 10 = 0$

$\therefore s = \dfrac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm 3i$

$\therefore$ the roots of the characteristic polynomial are $-2, -1+3i, -1-3i$ and they are all in the left hand plane $\rightarrow \therefore$ the system is stable

3] 2.



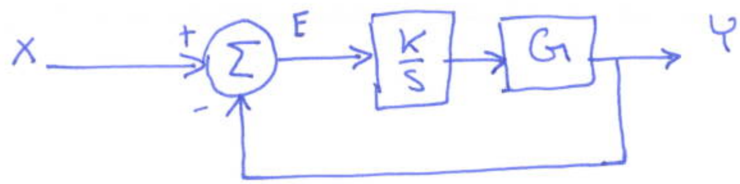
$$\frac{E}{X} = \frac{1}{1+KG} = \frac{1}{1+K\left[\dfrac{s^2+12s+10}{s^3+4s^2+14s+20}\right]} = \frac{s^3+4s^2+14s+20}{s^3+s^2(4+K)+s(14+12K)+(20+10K)}$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\cdot\frac{E}{X}\cdot X = \lim_{s\to 0} s\cdot\left[\quad\right]\times\frac{1}{s} = \frac{20}{20+10K}$$

for the steady state error to be zero, K has to be equal to ∞!
so the closed loop unity feedback system cannot track a step input using only a proportional controller.

Alternatively: $KG(s)$ has no roots at $s=0$ $\rightarrow$ $\therefore$ This is a type 0 system
$\therefore$ the system cannot track a step input with zero error

3] 3. An integral controller would eliminate the steady state erros:



$$\frac{E}{X} = \frac{1}{1+GH} = \frac{1}{1+\frac{K}{S}G} = \frac{S^4+4S^3+14S^2+20S}{S^4+4S^3+(14+K)S^2+(20+12K)S+10K}$$

. check the range of K that guarantees the stability of the system:

apply Routh criterion:

$$\begin{array}{llll} S^4 & 1 & 14+K & 10K \\ S^3 & 4 & 20+12K & \\ S^2 & 9-2K & 10K & \\ S^1 & \frac{-24K^2+28K+180}{9-2K} & & \\ S^0 & 10K & & \end{array}$$

$\therefore$ ① $10K>0 \rightarrow K>0$

② $9-2K>0 \rightarrow K<9/2$

③ $\frac{-24K^2+28K+180}{9-2K}>0 \rightarrow$ from ② $\rightarrow -24K^2+28K+180>0$

$\therefore\ 6K^2-7K-45<0 \rightarrow \therefore\ -2.22<K<3.38$

$\therefore\ 0<K<3.38$

choose $K=1$:

$$\lim_{t\to\infty} e(t) = \lim_{S\to 0} SE(s) = \lim_{S\to 0}\left[S \times \frac{S^4+4S^3+14S^2+20S}{S^4+4S^3+15S^2+34S+10} \times \frac{1}{S}\right]$$

$$= 0$$