ECE 141 - Feedback Control
Midterm Solutions

Problem (1 - Bungee Jumping)**.**

Solution (**Vers. A & B**)**.** Construct an ODE modeling the dynamics of the jumper via Newton's Laws.

$$m\ddot{x}(t) = -kx(t) - b\dot{x}(t) + mg \cdot \theta(t)$$

$m > 0$ is the mass of the jumper, $k > 0$ is the elastic constant of the cord, $b > 0$ is the coefficient of aerodynamic drag, and mg is the force of gravity (that acts on the jumper after $t = 0$), and θ is the Heaviside Step Function, i.e. $\theta(t \geq 0) = 1$ and $\theta(t < 0) = 0$.

Assuming zero initial conditions, i.e. the jumper initiates with zero velocity on the top of the bridge or cliff in the depicted coordinate frame, we take the Laplace Transform of the ODE.

$$ms^2X(s) = -kX(s) - bsX(s) + \frac{mg}{s} \implies X(s) = \frac{mg}{s\,(ms^2 + bs + k)}$$

$sX(s)$ has stable poles, because $X(s)$ has stable poles except one marginally-stable pole at zero. Indeed, given $b > 0$ and via the quadratic formula, the roots of the characteristic polynomial $p(s) = ms^2 + bs + k$ that excludes the marginally-stable pole at zero are $\left\{\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}\right\} \subset \mathbb{R}_{<0} + i\mathbb{R}$. Equivalently, you can compute the Routh Array and apply the Routh Criterion to deduce stability.

$$\begin{array}{c|cc} s^2 & m & k \\ s^1 & b & \\ s^0 & k & \end{array} \qquad m, b, k > 0 \implies \text{s X(s) is stable!}$$

Necessarily, $s^2X(s)$ also has stable poles since $sX(s)$ has stable poles. Applying the Final Value Theorem on both $X(s)$ and $sX(s)$, we deduce that:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{mg}{ms^2 + bs + k} = \frac{mg}{k}$$
$$\lim_{t\to\infty} \dot{x}(t) = \lim_{s\to 0} s \cdot sX(s) = \lim_{s\to 0} \frac{mgs}{ms^2 + bs + k} = 0$$

In other words, the position of the jumper converges to the position such that the elastic tension counteracts the force of gravity, i.e. $kx = mg$. Moreover, the velocity of the jumper also converges to zero due to the non-conservative kinetic energy sink of the aerodynamic drag. Hence, the state $(x, \dot{x}) = (mg/k, 0)$ is a stable equilibrium state of the system.

Assume we have control over the design of b and k, and want to design the frequency response of the dynamics of the jumper. In this case, we can ignore the pole at zero, because the frequency response of the system is invariant of constants. In other words, applying a Partial Fraction Decomposition separates the poles into a constant step with a pole at zero and an oscillatory component with complex poles, which inverts to shifted sinusoidal functions that have the same frequency response as their un-shifted variants modulo the zero-frequency term of the Fourier Series. Hence, it suffices to design and analyze only the complex poles of the system. (Note that other types of inputs besides the step function can non-trivially affect the frequency response, but that's not relevant here.)

$$X(s) = \frac{mg}{s\left(ms^2 + bs + k\right)} = \frac{g}{s\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)}$$

Utilizing our knowledge of transfer functions of second-order systems, we deduce that:

$$\omega_n^2 = \frac{k}{m} \implies \omega_n = \sqrt{\frac{k}{m}} \qquad \sigma = \omega_n\zeta = \frac{b}{2m} \implies \zeta = \frac{b}{2\sqrt{mk}}$$

To design the system to oscillate (and necessarily remain stable), we would need to have $\zeta \in (0, 1)$.

$$\zeta = \frac{b}{2\sqrt{mk}} \in (0, 1) \implies 0 < b < 2\sqrt{mk}$$

To design the system to have no oscillations (and necessarily remain stable), we would need to have $\zeta \geq 1$.

$$\zeta = \frac{b}{2\sqrt{mk}} \geq 1 \implies b \geq 2\sqrt{mk}$$

Both of these inequalities could have been deduced via analyzing the discriminant term of the quadratic formula for the characteristic polynomial of the system.

To maximize the amount of time spent oscillating with large amplitude and high frequency, we can take sufficiently small aerodynamic drag b or sufficiently large elastic constant k. Both will decrease ζ and rotate the complex poles closer to the imaginary axis, which simultaneously reduces the exponential decay of the dynamics while increasing the frequency of the oscillation. An inferior answer would be to consider the settling time $t_s = 4.6/\sigma$, because while it maximizes the time spent oscillating with non-trivial amplitude, it does not consider or improve the frequency of oscillation due to σ being the $\mathbb{R}$-component of the complex poles of the system, and induces a design policy solely on b. You could also argue that increasing m would decrease ζ (and decrease σ), but there is a trade-off in frequency versus exponential decay in adjusting $\omega_n = \sqrt{k/m}$, since decreasing ω_n decreases both the exponential decay (desirable) and the frequency (undesirable) of the response.

Assume the force of gravity is a control input and compute the transfer function $T(s)$ from mg/s to X.

$$X(s) = \frac{mg}{s\left(ms^2 + bs + k\right)} = \frac{1}{ms^2 + bs + k} \cdot \frac{mg}{s} \implies T(s) = \frac{1}{ms^2 + bs + k}$$

To set $t_s \leq 46/5$ and $t_r \leq 3/5$, we would have:

$$t_s = \frac{4.6}{\sigma} = \frac{9.2m}{b} \leq \frac{46}{5} \implies b \geq m$$
$$t_r = \frac{1.8}{\omega_n} = 1.8 \cdot \sqrt{\frac{m}{k}} \leq \frac{3}{5} \implies k \geq 9m$$

To set $t_s \leq 23/5$ and $t_r \leq 9/5$, we would have:

$$t_s = \frac{4.6}{\sigma} = \frac{9.2m}{b} \leq \frac{23}{5} \implies b \geq 2m$$
$$t_r = \frac{1.8}{\omega_n} = 1.8 \cdot \sqrt{\frac{m}{k}} \leq \frac{9}{5} \implies k \geq m$$

Solution (**Vers. C & D**). Construct an ODE modeling the dynamics of the jumper via Newton's Laws. Assuming that the coordinate frame of the jumper is calibrated to zero where the force of gravity is symmetric to elastic tension, we deduce that:

$$m\ddot{x}(t) = -kx(t) - b\dot{x}(t)$$

Necessarily, $m > 0$ is the mass of the jumper, $k > 0$ is the elastic constant of the cord, and $b > 0$ is the coefficient of aerodynamic drag.

Assuming zero initial position $x(0) = 0$ and initial velocity $\dot{x}(0) = v_0$, i.e. the jumper accelerates to initial velocity v_0 via an impulse of momentum, we take the Laplace Transform of the ODE.

$$m\left[s^2X(s) - s\cdot \cancelto{0}{x(0)} - \dot{x}(0)\right] = -kX(s) - b\left[sX(s) - \cancelto{0}{x(0)}\right] \implies X(s) = \frac{mv_0}{ms^2 + bs + k}$$

$sX(s)$ has stable poles, because $X(s)$ has stable poles. Indeed, given $b > 0$ and via the quadratic formula, the roots of the characteristic polynomial $p(s) = ms^2 + bs + k$ are $\left\{\dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}\right\} \subset \mathbb{R}_{<0} + i\mathbb{R}$. Equivalently, you can compute the Routh Array and apply the Routh Criterion to deduce stability.

$$\begin{array}{c|cc} s^2 & m & k \\ s^1 & b & \\ s^0 & k & \end{array} \qquad m, b, k > 0 \implies \text{X(s) is stable!}$$

Necessarily, $s^2X(s)$ also has stable poles since $sX(s)$ has stable poles. Applying the Final Value Theorem on both $X(s)$ and $sX(s)$, we deduce that:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\cdot X(s) = \lim_{s\to 0} \frac{mv_0 s}{ms^2 + bs + k} = 0$$
$$\lim_{t\to\infty} \dot{x}(t) = \lim_{s\to 0} s\cdot sX(s) = \lim_{s\to 0} \frac{mv_0 s^2}{ms^2 + bs + k} = 0$$

Necessarily, the position of the jumper converges to zero, which is precisely where the elastic tension counteracts the force of gravity such that the cord is *relatively* unstretched in the assumed stretched coordinate frame of the jumper. Moreover, the velocity of the jumper also converges to zero due to the non-conservative kinetic energy sink of the aerodynamic drag. Hence, the state $(x, \dot{x}) = (0, 0)$ is a stable equilibrium state of the system.

Assume we have control over the design of b and k, and want to design the frequency response of the dynamics of the jumper.

$$X(s) = \frac{mv_0}{ms^2 + bs + k} = \frac{v_0}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

Utilizing our knowledge of transfer functions of second-order systems, we deduce that:

$$\omega_n^2 = \frac{k}{m} \implies \omega_n = \sqrt{\frac{k}{m}} \qquad \sigma = \omega_n\zeta = \frac{b}{2m} \implies \zeta = \frac{b}{2\sqrt{mk}}$$

To design the system to oscillate (and necessarily remain stable), we would need to have $\zeta \in (0, 1)$.

$$\zeta = \frac{b}{2\sqrt{mk}} \in (0, 1) \implies 0 < b < 2\sqrt{mk}$$

To design the system to have no oscillations (and necessarily remain stable), we would need to have $\zeta \geq 1$.

$$\zeta = \frac{b}{2\sqrt{mk}} \geq 1 \implies b \geq 2\sqrt{mk}$$

Both of these inequalities could have been deduced via analyzing the discriminant term of the quadratic formula for the characteristic polynomial of the system.

To maximize the amount of time spent oscillating with large amplitude and high frequency, we can take sufficiently small aerodynamic drag b or sufficiently large elastic constant k. Both will decrease ζ and rotate the complex poles closer to the imaginary axis, which simultaneously reduces the exponential decay of the dynamics while increasing the frequency of the oscillation. An inferior answer would be to consider the settling time $t_s = 4.6/\sigma$, because while it maximizes the time spent oscillating with non-trivial amplitude, it does not consider or improve the frequency of oscillation due to σ being the $\mathbb{R}$-component of the complex poles of the system, and induces a design policy solely on b. You could also argue that increasing m would decrease ζ (and decrease σ), but there is a trade-off in frequency versus exponential decay in adjusting $\omega_n = \sqrt{k/m}$, since decreasing ω_n decreases both the exponential decay (desirable) and the frequency (undesirable) of the response.

Assume that the initial impulse momentum mv_0 is a control input to the system and compute the transfer function $T(s)$ from mv_0 to X.

$$X(s) = \frac{mv_0}{ms^2 + bs + k} = \frac{1}{ms^2 + bs + k} \cdot mv_0 \implies T(s) = \frac{1}{ms^2 + bs + k}$$

To set $t_s \leq 2/5$ and $t_r \leq 1/5$, we would have:

$$t_s = \frac{4.6}{\sigma} = \frac{9.2m}{b} \leq \frac{2}{5} \implies b \geq 23m$$
$$t_r = \frac{1.8}{\omega_n} = 1.8 \cdot \sqrt{\frac{m}{k}} \leq \frac{1}{5} \implies k \geq 81m$$

To set $t_s \leq 46/5$ and $t_r \leq 18/5$, we would have:

$$t_s = \frac{4.6}{\sigma} = \frac{9.2m}{b} \leq \frac{46}{5} \implies b \geq m$$
$$t_r = \frac{1.8}{\omega_n} = 1.8 \cdot \sqrt{\frac{m}{k}} \leq \frac{18}{5} \implies k \geq \frac{m}{4}$$

Problem (2 - Control Systems Design)**.**

Solution. Version D

On simplifying the block diagram, we can see that the system is composed of a transfer function in series with a unity feedback transfer function.

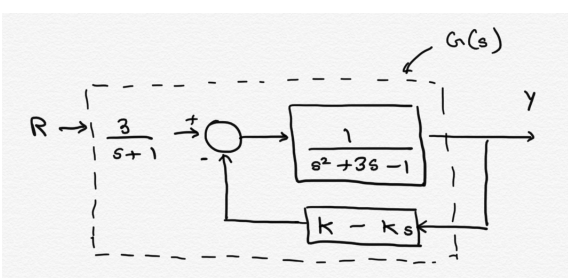

Figure 1: Simplified Diagram

Transfer function from R to Y would be given by:

$$T(s) = \frac{3}{s^3 + (4-K)s^2 + 2s + K - 1}$$

Applying Routh–Hurwitz Criteria, we get the following Table,

Routh-Hurwitz		
s^3	1	2
s^2	4-K	K-1
s^1	$\frac{9-3K}{4-K}$	0
s	K-1	0

Therefore for system stability, we need

$$K \in (1, 3)$$

The transfer function for the error in case of step input is given by

$$E(s) = \frac{1}{s} \times \frac{s^3 + (4-K)s^2 + 2s + K - 4}{s^3 + (4-K)s^2 + 2s + K - 1}$$

For $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$, we can find that $K = 4$, if we want the steady state error to be zero. However this is not permissible as this would make the system to be unstable.

We know that, an integral controller helps drive down the steady state error to zero. Therefore we add an integral controller to the system as shown below:

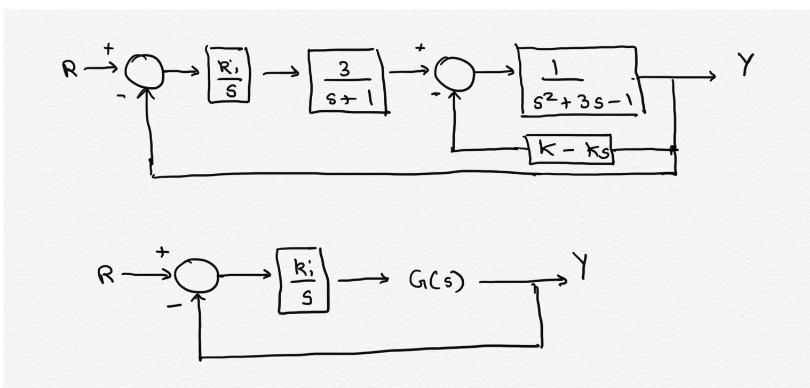

Figure 2: Integral Controller added to the system

On adding the integral controller, the transfer function for the error can be given by:

$$E(s) = R(s) \times \frac{s^4 + (4-K)s^3 + 2s^2 + (K-1)s}{s^4 + (4-K)s^3 + 2s^2 + (K-1)s + 3K_i}$$

We can see that the characteristic equation in this case is : $s^4 + (4-K)s^3 + 2s^2 + (K-1)s + 3ki$

Before we move on to finding the steady-state error, we need to check if the system is stable or not. For this, we can compute the Routh-Hurwitz criteria and you should get the first column as follows:

First Column of Routh-Hurwitz Criteria
1
$4 - K$
$(3K - 9)/(K - 4)$
$(K^2 - 4K + 3)/(K - 3) + (K_i(K^2 - 8K + 16))/(K - 3)$
$3K_i$

From the above table, we can see that we can choose some values for K and K_i such that the first column has all positive values. So for example, if we choose $K = 2$ and $ki = 0.2$, then the first column has all positive values and therefore the resulting system would be stable. So we can find controller gains such that the system is stable. Now we can proceed to find the steady-state error of the system.

For a step-input, the error transfer function is:

$$E(s) = \frac{1}{s} \times \frac{s^4 + (4 - K)s^3 + 2s^2 + (K - 1)s}{s^4 + (4 - K)s^3 + 2s^2 + (K - 1)s + 3K_i}$$

Applying the theorem, $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$, we can see that the steady state error is zero for the values of K and K_i we have computed above, which is what we wanted. You need to show the existence of valid value of K and K_i before calculating the steady state error.

For all the other versions, the logic above remains the same. So we have just mentioned the final transfer functions and answers that you'll get.

Solution. Version C
Transfer function from R to Y would be given by:

$$T(s) = \frac{-6}{(K - 5)s^3 + (2K - 12)s^2 + (-K - 1)s - 2K + 6}$$

For system stability, we need

$$K \in (3, 5)$$

Transfer function for E(s) is:

$$E(s) = \frac{1}{s} \times \frac{(K - 5)s^3 + (2K - 12)s^2 + (-K - 1)s - 2K + 12}{(K - 5)s^3 + (2K - 12)s^2 + (-K - 1)s - 2K + 6}$$

For steady-state error to be zero, we need $K = 6$, which is not permissible.

On adding a integral controller, the transfer function for the error can be given by:

$$E(s) = \frac{1}{s} \times \frac{(5 - K)s^4 + (12 - 2K)s^3 + (K + 1)s^2 + (2K - 6)s}{(5 - K)s^4 + (12 - 2K)s^3 + (K + 1)s^2 + (2K - 6)s + 6K_i}$$

For estimating the stability, we compute the Routh-Hurwitz criteria and get the following:

First Column of Routh-Hurwitz Criteria
$5 - K$
$12 - 2K$
$(3K - 21)/(K - 6)$
$(2(K^2 - 10K + 21))/(K - 7) + (2K_i(2K^2 - 24K + 72))/(K - 7)$
$6K_i$

From the table above we can calculate a value of K and K_i such that the system is stable. For example, if we choose $K = 4$ and $K_i = 1/6$, we can see that the resulting system is stable.

Using these values for K and K_i, if we compute the steady state error, we can see that it is zero.

Solution. Version B
Transfer function from R to Y would be given by:

$$T(s) = \frac{-(s+3)}{s^3 + (K+2)s^2 + (4K-4)s + 3K - 5}$$

For system stability, we need

$$K \in (5/3, \infty)$$

Transfer function for E(s) is:

$$E(s) = \frac{1}{s} \times \frac{s^3 + (K+2)s^2 + (4K-3)s + 3K - 2}{s^3 + (K+2)s^2 + (4K-4)s + 3K - 5}$$

For steady-state error to be zero, we need $K = 2/3$, which is not permissible.

On adding a integral controller, the transfer function for the error can be given by:

$$E(s) = \frac{1}{s} \times \frac{s^4 + (K+2)s^3 + (4K-4)s^2 + (3K-5)s}{s^4 + (K+2)s^3 + (4K-4)s^2 + (3K - K_i - 5)s - 3K_i}$$

For estimating the stability, we compute the Routh-Hurwitz criteria and get the following:

First Column of Routh-Hurwitz Criteria
1
$K + 2$
$K_i/(K+2) + (4K^2 + K - 3)/(K+2)$
$-(14K + 17K^2 - 12K^3 + K_i^2 - K_i(K^2 + 14K + 10) - 15)/(4K^2 + K + K_i - 3)$
$-3K_i$

From the table above we can calculate a value of K and K_i such that the system is stable. For example, if we choose $K = 2$ and $K_i = -1/12$, we can see that the resulting system is stable.

Using these values for K and K_i, if we compute the steady state error, we can see that it is zero.

Solution. Version A
Transfer function from R to Y would be given by:

$$T(s) = \frac{-4}{Ks^3 + (5K+1)s^2 + (4K+6)s - 4K + 8}$$

For system stability, we need

$$K \in (-1/5, 2)$$

Transfer function for E(s) is:

$$E(s) = \frac{1}{s} \times \frac{Ks^3 + (5K+1)s^2 + (4K+6)s - 4K + 12}{Ks^3 + (5K+1)s^2 + (4K+6)s - 4K + 8}$$

For steady-state error to be zero, we need $K = 3$, which is not permissible.

On adding a integral controller, the transfer function for the error can be given by:

$$E(s) = \frac{1}{s} \times \frac{Ks^4 + (5K+1)s^3 + (6+4K)s^2 + (8-4K)s}{Ks^4 + (5K+1)s^3 + (6+4K)s^2 + (8-4K)s - 4K_i}$$

For estimating the stability, we compute the Routh-Hurwitz criteria and get the following:

First Column of Routh-Hurwitz Criteria
K
$5K + 1$
$(24K^2 + 26K + 6)/(5K + 1)$
$((2(25K^2 + 10K + 1))/(12K^2 + 13K + 3))K_i + (2(-24K^3 + 22K^2 + 46K + 12))/(12K^2 + 13K + 3)$
$-4K_i$

From the table above we can calculate a value of K and K_i such that the system is stable. For example, if we choose $K = 1$ and $K_i = -1$, we can see that the resulting system is stable.

Using these values for K and K_i, if we compute the steady state error, we can see that it is zero.