ECE141, Principles of Feedback Systems **Name:** Solutions
UCLA Winter 2019
Midterm Exam
02/12/2019
Time Limit: 2 hours

(a) This exam booklet contains 14 pages (including this cover page) and 5 problems. Total of points is 100.

(b) All books and notes must remain closed. You are allowed to have a single double-sided letter-sized cheat sheet.

(c) Simple calculators are allowed. But no fancy calculators, smartphones, or any other smart and/or connected devices.

(d) Please fully justify your answers and clearly show all the intermediate steps in all your solutions. And, when appropriate, box your final answer.

(e) Please do NOT write your answers on the back of any pages. Answers written on the back will NOT be graded.

(f) **Good Luck...**

Grade Table (for instructor use only)

Question	Points	Score
1	10	
2	24	
3	24	
4	22	
5	20	
Total:	100	

1. (10 points) **Quick Review**

   Carefully read each statement below and identify it as *True* or *False* by clearly writing **T** or **F** in the box.

   - T ☑ The BIBO stability of an LTI system is equivalent to the asymptotic stability in Lyapunov sense of the equilibrium point at the origin $X = 0$.
   - F ☒ A stable closed-loop system can track a ramp input with zero steady-state error as long as it has at least one pure integrator in the loop.
   - F ☒ Final Value Theorem can be used to find the steady-state response of a marginally stable system.
   - F ☒ Any LTI system has a unique state-space representation.
   - T ☑ A lag compensator will reduce steady-state error.
   - T ☑ A Proportional-Integral (PI) compensator is a lag compensator whereas a Proportional-Derivative (PD) compensator is a lead compensator.
   - F ☒ A unity feedback system with a non-minimum phase loop transfer function will always remain stable for large gains.
   - T ☑ As a pole of a 2nd-order system moves *closer* to the $j\omega$ axis, while its imaginary part remains constant, the settling time of the step response gets larger.
   - F ☒ An unstable pole in a plant can be canceled by a zero in the compensator as long as the transient performance remains acceptable.
   - T ☑ Making $|L(j\omega)| \ll 1$, where $L(s)$ is the loop transfer function, would improve the robust stability of a feedback system.

2. (24 points) **Linearization, State-Space**
Predator-prey models are commonly applied to ecological systems where there are two dominant species, one of which acts as herbivorous prey and the other as a carnivorous predator. Predator-prey systems have been studied for decades and are known to exhibit interesting dynamics. In this case, the system being modeled is the interaction of wolves and deer in an Alaskan wildlife preserve. The model is given as follows:

$$\dot{d} = (30 - w)d \tag{1}$$
$$\dot{w} = (0.5d - 40)w - f \tag{2}$$

where d is the population of deer, w is the population of wolves, and f represents the control input used by conservation ecologists to control the population of wolves in order to ensure the health of the overall ecological system.

(a) (5 points) What will be the population of deer and wolves in the equilibrium with no input? i.e., with $f = 0$, find out the possible equilibrium points (d_e, w_e) for this nonlinear system.

At Equilibrium, we have : $\dot{\underline{x}} = \begin{bmatrix} \dot{d} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} (30 - W)d = 0 \longrightarrow \begin{cases} d_e = 0 \\ w_e = 30 \end{cases} \\ (0.5d - 40)W = 0 \longrightarrow \begin{cases} d_e = 80 \\ w_e = 0 \end{cases} \end{cases}$$

$\Rightarrow$ equilibrim point #1 : $(d_e, w_e) = (0, 0)$
equilibrium point #2 : $(d_e, w_e) = (80, 30)$

(b) (8 points) Linearize the model around the **non-zero** equilibrium point, and find matrices A and B such that $\delta\dot{\mathbf{x}} = A\delta\mathbf{x} + B\delta f$ where $\mathbf{x} \triangleq [d \quad w]^T$ and $\delta\mathbf{x} \triangleq \mathbf{x} - [d_e \quad w_e]^T$.

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \longrightarrow \delta\dot{\underline{x}} = A\,\delta\underline{x} + B\,\delta\underline{u}$$

where: $A = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{(\underline{x}_e, \underline{u}_e)}$ and $B = \left.\frac{\partial \underline{f}}{\partial \underline{u}}\right|_{(\underline{x}_e, \underline{u}_e)}$

$$\Rightarrow \dot{\underline{x}} = \begin{bmatrix} \dot{d} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} (30-w)d \\ (0.5d-40)w - f \end{bmatrix}$$

$$\Rightarrow A = \frac{\partial \underline{f}}{\partial \underline{x}} = \left.\begin{bmatrix} \frac{\partial}{\partial d}[(30-w)d] & \frac{\partial}{\partial w}[(30-w)d] \\ \frac{\partial}{\partial d}[(0.5d-40)w] & \frac{\partial}{\partial w}[(0.5d-40)w] \end{bmatrix}\right|_{(80,30)}$$

$$\Rightarrow A = \left.\begin{bmatrix} 30-w & -d \\ 0.5w & 0.5d-40 \end{bmatrix}\right|_{(80,30)} = \begin{bmatrix} 0 & -80 \\ 15 & 0 \end{bmatrix}$$

$$B = \left[\frac{\partial}{\partial f}[(30-w)d] \quad \frac{\partial}{\partial f}[(0.5d-40)w - f]\right]^T = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(c) (5 points) Determine if the linearized system you found in Part (b) is stable, unstable, or marginally stable.

linearized system: $\begin{bmatrix} \delta\dot{d} \\ \delta\dot{w} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -80 \\ 15 & 0 \end{bmatrix}}_{A}\begin{bmatrix} \delta d \\ \delta w \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{B}\delta f$

The poles of the system are the eigenvalues of A.

$$\det(sI - A) = \begin{vmatrix} s & 80 \\ -15 & s \end{vmatrix} = s^2 + 1200 = 0 \Rightarrow s_{1,2} = \pm j20\sqrt{3}$$

$\Rightarrow$ marginally stable since poles are on the $j\omega$ axis.

(d) (6 points) Assuming the deviation of the deer population from its equilibrium δd as the output, find the Transfer Function from the input δf to the output δd in the linearized system.

$$\begin{bmatrix} \delta \dot{d} \\ \delta \dot{w} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & -80 \\ 15 & 0 \end{bmatrix}}^{A} \begin{bmatrix} \delta d \\ \delta w \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{B} \delta f$$

$$\delta y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} \delta d \\ \delta w \end{bmatrix}$$

$$\Longrightarrow H(s) = \frac{\delta Y(s)}{\delta F(s)} = C(sI-A)^{-1}B$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 80 \\ -15 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{s^2+1200} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -80 \\ 15 & s \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \frac{80}{s^2+1200}$$

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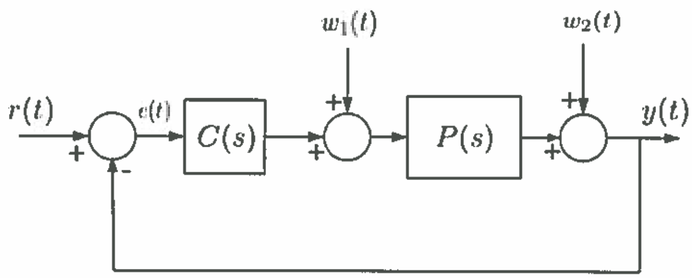
3. (24 points) **Feedback Fundamentals, Transfer Function Properties, IMP**
Consider the following feedback system:



Assume the following transfer functions for the plant and the compensator:

$$P(s) = \frac{1}{s+10}, \quad \text{and} \quad C(s) = \frac{1}{s}$$

(a) (6 points) In the s-domain, write $Y(s)$ in terms of the system inputs $R(s)$, $W_1(s)$, and $W_2(s)$

$$Y(s) = \frac{PC}{1+PC} R(S) + \frac{P}{1+PC} W_1(S) + \frac{1}{1+PC} W_2(s)$$

$$\Longrightarrow Y(s) = \frac{\frac{1}{s(s+10)}}{1+\frac{1}{s(s+10)}} R(S) + \frac{\frac{1}{s+10}}{1+\frac{1}{s(s+10)}} W_1(S) + \frac{1}{1+\frac{1}{s(s+10)}} W_2(s)$$

$$\Longrightarrow Y(s) = \frac{1}{s^2+10s+1} R(s) + \frac{s}{s^2+10s+1} W_1(s) + \frac{s^2+10s}{s^2+10s+1} W_2(s)$$

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(b) (4 points) Assuming zero initial conditions, write an Ordinary Differential Equation that relates the output $y(t)$ to the inputs $r(t)$, $w_1(t)$, and $w_2(t)$.

$$\Longrightarrow (s^2+10s+1)Y(s) = R(s) + sW_1(s) + (s^2+10s)W_2(s)$$

Assuming zero initial conditions:

$$\Longrightarrow \ddot{y}(t) + 10\dot{y}(t) + y(t) = r(t) + \dot{w}_1(t) + \ddot{w}_2(t) + 10\dot{w}_2(t)$$

(c) (7 points) Assume the following reference and disturbance inputs ($u(t)$ is the unit step function):

$$r(t) = (10 + \sin(t))u(t) \quad (3)$$
$$w_1(t) = 10u(t) \quad (4)$$
$$w_2(t) = \cos(1000t)u(t) \quad (5)$$

Find the the closed-loop system response in the steady-state $y_{ss}(t)$.

Using the fundamental property of transfer functions, we can write:

$$y_{ss}(t) = H_{ry}(0).10 + |H_{ry}(j1)| \sin(t + \measuredangle H_{ry}(j1))$$
$$+ H_{w_1y}(0).10 + |H_{w_2y}(j1000)| \cos(1000t + \measuredangle H_{w_2y}(j1000))$$

$$H_{ry}(s) = \frac{1}{s^2+10s+1} \longrightarrow H_{ry}(0) = 1.0 \quad \& \quad H_{ry}(j1) = \frac{1}{j10} = -0.1j = 0.1e^{-j\frac{\pi}{2}}$$

$$H_{w_1y}(s) = \frac{s}{s^2+10s+1} \longrightarrow H_{w_1y}(0) = 0$$

$$H_{w_2y}(s) = \frac{s^2+10s}{s^2+10s+1} \longrightarrow H_{w_2y}(j1000) \simeq 1.0 = 1.0e^{j0.0}$$

$$\Longrightarrow y_{ss}(t) = 10 + 0.1\sin(t - \tfrac{\pi}{2}) + \cos 1000t$$

(d) (3 points) Can this closed-loop system track step reference input with zero steady-state error? i.e., Will we have $e(t) \to 0$ as $t \to \infty$ when $r(t) = u(t)$ and $w_1(t) = w_2(t) = 0$? If yes, why? If not, why not? Please explain. No need to calculate.

Yes, the closed-loop system is stable and the loop has a pure integrator. So, from IMP, we know that the closed-loop system can indeed track a step reference input with zero steady-state error.

(e) (4 points) Can this closed-loop system reject a ramp disturbance at the input to the plant in the steady-state? i.e., Will we have $y(t) \to 0$ as $t \to \infty$ for $w_1(t) = tu(t)$ and $r(t) = w_2(t) = 0$? If yes, why? If no, how would you propose to change the compensator $C(s)$ to make that happen?

No. From IMP, we know that $C(s)$ would need at least two pure integrators to reject a ramp disturbance (that has two unstable poles at $s=0$) at the input to the plant. Now, if we set $C(s) = \frac{K}{s^2}$, the closed-loop system cannot be stable:

$1 + KL(s) = 0 \to 1 + \frac{K}{s^2} \cdot \frac{1}{s+10} = s^3 + 10s^2 + K = 0$

$\to$ coefficient of s term is zero $\to$ unstable.

So instead, similar to the HW4 problem, we can add a zero too, or effectively add a PI compensator to our pure integrator, i.e. $C(s) = (K_p + \frac{K_I}{s}) \cdot \frac{1}{s}$

$\Rightarrow 1 + KL(s) = 1 + (K_p + \frac{K_I}{s}) \frac{1}{s} \frac{1}{s+10} = 0 \to$

$\to s^3 + 10s^2 + K_p s + K_I = 0$

$$\begin{array}{c|cc} s^3 & 1 & K_p \\ s^2 & 10 & K_I \\ s^1 & \frac{10K_p - K_I}{10} & 0 \\ s^0 & K_I & \end{array}$$

$\to$ closed-loop will be stable as long as: $K_I > 0$ and $K_I < 10K_p$

(In the exam, you did not need to form the Routh-Hurwitz array and prove stability)

4. (22 points) **Plotting Root Locus**
The primary mirror of a large telescope can have many smaller hexagonal segments with the orientation of each segment actively controlled. Suppose the unity feedback system for the mirror segments has the following Loop Transfer Function:

$$L(s) = \frac{1}{s(s^2 + 2s + 5)}$$

We want to plot the root locus of the characteristic equation $1 + KL(s) = 0$ for $K \geq 0$:

(a) (1 point) Determine the number of branches in the Root Locus: .3..(3 poles)

(b) (2 points) Determine the portion of the real axis on the root locus: ....

$Re(s) < 0$

(c) (3 points) Determine the angle of each asymptote: $\theta_a = \ldots\ldots$

$$\theta_a = \frac{(2l+1)180}{n-m} = \frac{(2l+1)180}{3-0} = 60, 180, 300$$

$l = 0, 1, 2$

(d) (3 points) Determine the asymptote centroid, i.e., the real axis intercept of the asymptotes: $\sigma_a = \ldots\ldots$

$$\sigma_a = \frac{\sum p - \sum z}{n-m} = \frac{-1-1}{3-0} = -\frac{2}{3}$$

(e) (4 points) Determine the angle of departure from the open-loop pole at $s = -1+2j$: (*note*: $\tan^{-1}(0.5) = 26.56°$)

Consider a point very close to the open-loop pole at $-1+2j$ and use the angles condition:

$$\varphi_{dep} = \sum \text{zero-angles} - \sum \text{pole-angles} - (2l+1)180$$

$$\varphi_{p_1} = 90° + \tan^{-1}\left(\frac{1}{2}\right) = 116.56°$$

$$\varphi_{p_2} = 90°$$

$$\Longrightarrow \varphi_{dep} = 0 - 116.56 - 90 - 180 = -386.56 = -26.56°$$

(f) (3 points) Using all of the above information, plot the root locus on the graph in the next page. Use arrows to show the direction of increasing gain.

(g) (6 points) Using Routh-Hurwitz method, estimate the minimum gain that would make the closed-loop system unstable. Then find out the points at which the Root Locus crosses the jw axis.

$$1 + K L(s) = 0$$

$$\Longrightarrow 1 + K \cdot \frac{1}{s(s^2+2s+5)} = 0 \longrightarrow s^3 + 2s^2 + 5s + K = 0$$

s^3	1	5
s^2	2	K
s^1	$\frac{10-K}{2}$	0
s^0	K	

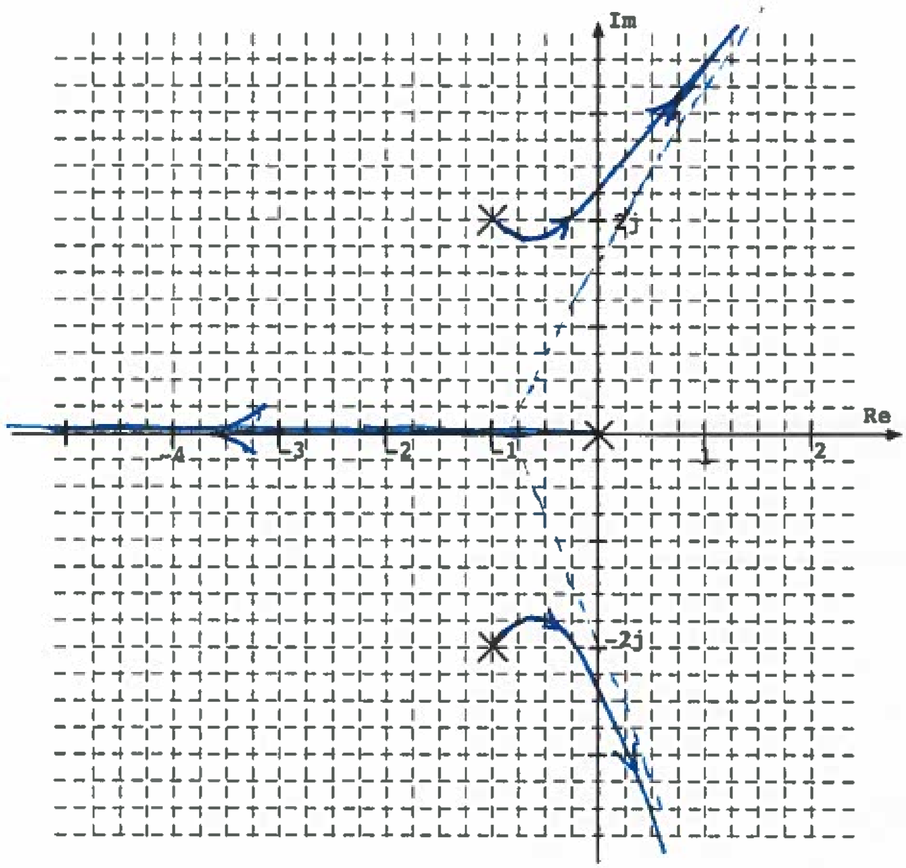
At $K=0$, there will be pole at $s=0$ and the ~~sy~~ closed-loop system would be unstable.

But from the RH array, it is clear that the minimum non-zero gain that would make the closed-loop system unstable, would be $\underline{K=10}$

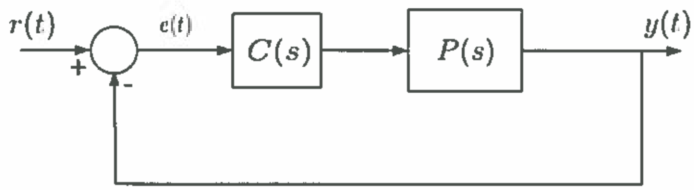
Then from the auxiliary equation from the RH table, we have:

$$2s^2 + K = 0 \longrightarrow 2s^2 + 10 = 0 \longrightarrow \underline{s_{1,2} = \pm j\sqrt{5}}$$

points of RL crossings with the jω axis


Im
Re
2j
-2j
-4
-3
-2
-1
1
2

5. (20 points) **Root Locus-based Design**
Consider the following feedback system:



And assume:

$$P(s) = \frac{1}{s(s+5)(s+500)}$$

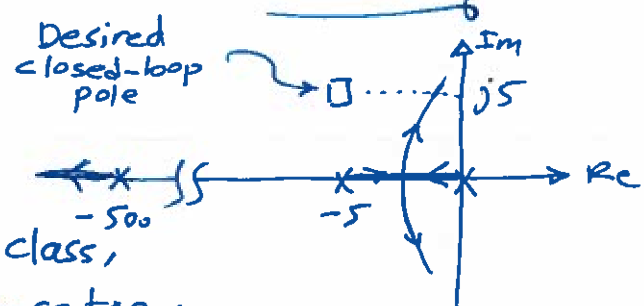
(a) (10 points) Using the Root Locus technique, design a compensator $C(s)$ such that the closed-loop system will have the maximum percentage overshoot $P.O.$ of 5% and the maximum 2% settling time t_s of 800 msec.

$P.O. = 5\% \Rightarrow \xi = 0.707 \Rightarrow \theta = 45^\circ$

$t_s = \frac{4}{\xi\omega_n} \Rightarrow \xi\omega_n = \frac{4}{0.8} = 5$

$\Big\}\rightarrow$ So to meet our transient specs, we can place the closed loop poles @ $-5 \pm j5$

It is clear from the RL for uncompensated system (right) that we cannot place the closed-loop pole at our desired location.



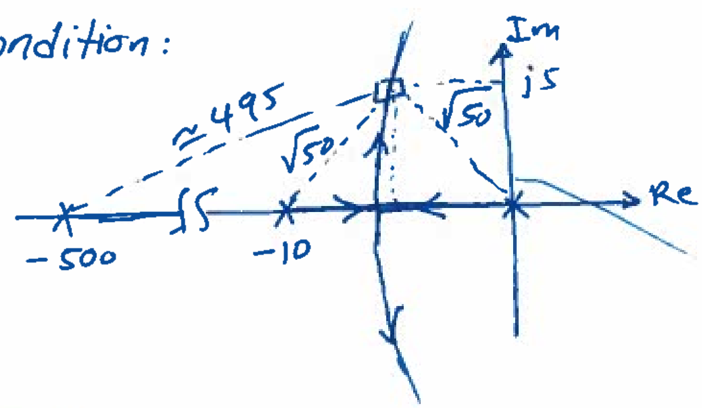
So, similar to the example in the class, we use the following lead compensator:

$$C(s) = K \cdot \frac{s+5}{s+10}$$

To find gain K, use magnitude condition:

$$K = \frac{\Pi \text{ pole lengths}}{\Pi \text{ zero lengths}}$$

$$= 495 \times \sqrt{50} \times \sqrt{50} = 24750$$



$$\Rightarrow C_{Lead}(s) = 24750 \cdot \frac{s+5}{s+10}$$

(b) (3 points) With the compensator you designed in Part (a), find the velocity error constant (a.k.a., ramp error constant) $K_v = \lim_{s\to 0} sL(s)$

$$K_v = \lim_{s\to 0} sL(s) = \lim_{s\to 0} \cancel{s} \times 24750 \times \frac{\cancel{s+5}}{s+10} \times \frac{1}{\cancel{s}(\cancel{s+5})(s+500)} =$$

$$= 24750 \times \frac{1}{10 \times 500} = \underline{4.95}$$

(c) (7 points) We have now been asked to reduce the closed-loop steady-state error to a ramp input by ten times, (i.e., to increase K_v ten times). Design a modified compensator which helps us achieve that while still meeting the transient performance specs in Part (a).

To increase K_v by 10 times, we can use a Lag compensator with $\frac{z}{p} = 10$. And because we do not want to disturb our already-placed closed-loop poles, we pick z and p both close to the $j\omega$ axis,

For example, $z = 0.1 \rightarrow p = 0.01$

$\rightarrow$ The overall Kompensator:

$$C(s) = C_{lead}(s)\, C_{lag}(s) = 24750 \cdot \frac{s+5}{s+10} \cdot \frac{s+0.1}{s+0.01}$$

If at home, perhaps w/ MATLAB, you find the closed-loop poles for the compensated system, you will see that the closed-loop poles will now be placed at: $s_1 \simeq -500$

$$s_2 \simeq -4.9 \pm j4.95$$

So it would have been ~~better~~ to place the poles a bit further inside the desired region s.t. after adding the Lag compensator, our transient specs would still be fully met!

Scratch Page