EE 141 Fundamentals of Feedback Control
Spring 2014

Midterm Examination
May 5, 2014

Instructions

1. No electronic aid is allowed during the examination period. Please put away all cellphones, tablets, laptops, calculators, etc. to where they are not accessible during the exam.
2. Show all your works and write clearly.
3. Indicate your final answers clearly.
4. Answer the questions in the spaces provided on the exam sheets. Should you require additional sheets of paper, please request the exam proctor.

Name: ______________________________________________ Section: ________

| Question | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 15 | |
| Total: | 50 | |

1. Consider the following state description of a nonlinear system:

$$\frac{dx_1}{dt} = (1 - x_1)(1 + x_2) - u(t)$$
$$\frac{dx_2}{dt} = (x_1 + x_2)^2 + u(t)$$

$$y(t) = x_1(t) + 2x_2(t)$$

(a) (5 points) Find an equilibrium state $x_{\text{eq}}$, given $u(t) = u_0(t) = 0$.

SOLUTION:
At an equilibrium,

$$f(t, x_{\text{eq}}, u_0) = 0.$$

Then,

$$(1 - x_1)(1 + x_2) = 0$$
$$(x_1 + x_2)^2 = 0$$

Therefore, $x_{\text{eq}} = (1, -1)$.

(b) (5 points) Linearize the system about $x_{\text{eq}}$ found from part (a). SOLUTION:

Want to obtain

$$\dot{\delta x} = A\delta x + Bu$$
$$y = C\delta x$$

Construct $A$:

$$A_{11} = \left.\frac{\partial f_1}{\partial x_1}\right|_{x=x_{\text{eq}}} = -1 - x_{\text{eq},2} = 0$$
$$A_{12} = \left.\frac{\partial f_1}{\partial x_2}\right|_{x=x_{\text{eq}}} = 1 - x_{\text{eq},1} = 0$$
$$A_{21} = \left.\frac{\partial f_2}{\partial x_1}\right|_{x=x_{\text{eq}}} = 2(x_{\text{eq},1} + x_{\text{eq},2}) = 0$$
$$A_{22} = \left.\frac{\partial f_2}{\partial x_2}\right|_{x=x_{\text{eq}}} = 2(x_{\text{eq},1} + x_{\text{eq},2}) = 0$$

$$\Rightarrow A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Construct $B$:

$$B_1 = \left.\frac{\partial f_1}{\partial u}\right|_{u=u_0} = -1$$
$$B_2 = \left.\frac{\partial f_2}{\partial u}\right|_{u=u_0} = 1$$

$$\Rightarrow B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Finally,

$$C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

Then,

$$\frac{d\delta x}{dt} = A\delta x(t) + B\delta u(t)$$
$$y(t) = C\delta x(t)$$

(c) (5 points) Show that the impulse reponse of the linearized system is a unit step.

SOLUTION: The transfer function (impulse response) is

$$\begin{aligned} \frac{Y(s)}{U(s)} &= C\,(sI - A)^{-1}\,B \\ &= -\frac{1}{s} + \frac{2}{s} = \frac{1}{s} \end{aligned}$$

Finally,

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = U(t),$$

where $U(t)$ is the unit step function.

2. Consider the unity feedback system with the open-loop transfer function

$$G(s) = \frac{10(s+1)}{s^3(s+10)(s+50)}$$

(a) (7 points) Determine the steady-state error for the unit step, unit ramp, and unit parabolic input.

SOLUTION:

Check stability first. The characteristic equation is

$$1 + G(s) = 0 \Rightarrow s^5 + 60s^4 + 500s^3 + 10s + 10 = 0$$

The coefficient for $s^2$ is 0. We need to check its Routh Array. The Routh Array is

$$\begin{array}{c|c|c|c} s^5 & 1 & 500 & 10 \\ s^4 & 60 & 0 & 10 \\ s^3 & 500 & \frac{590}{60} & \\ s^2 & -\frac{590}{500} & 10 & \\ s^1 & \vdots & & \\ s^0 & & & \end{array}$$

The Routh Array shows at least one sign change in the first column, which indicates that there exists at least one zero whose real part is positive. Therefore, the system is not stable, and there does not exist steady-state error.

(b) (3 points) Use the answers above to find the steady-state error for

i. $r(t) = -4U(t)$,

ii. $r(t) = 2tU(t)$,

iii. $r(t) = -t^2U(t)$,

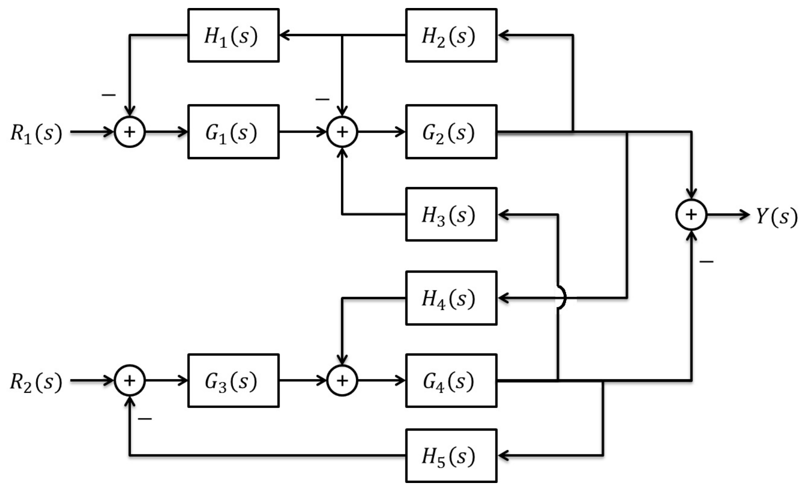
where

$$U(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

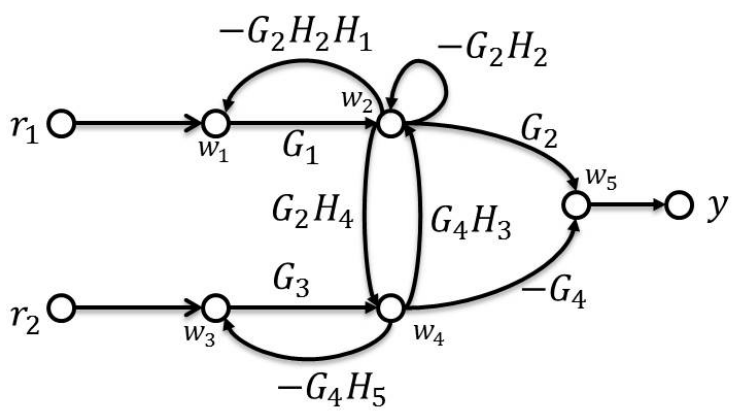
SOLUTION:

No steady-state error exists.

3. Consider the system described by the block diagram below:




(a) (5 points) Draw a signal graph for the given system. Indicate clearly your choice of node variables.



(b) (5 points) Use Mason's Rule to determine the transfer fuctions for $Y(s)/R_1(s)$ and $Y(s)/R_2(s)$.

SOLUTION:
Loops

$$\begin{aligned} L_1 &= -G_1G_2H_1H_2 \\ L_2 &= -G_2H_2 \\ L_3 &= -G_3G_4H_5 \\ L_4 &= G_2H_4G_4H_3 \end{aligned}$$

Non-touching loop pairs

$$\begin{aligned} &\{L_1, L_3\} \\ &\{L_2, L_3\} \end{aligned}$$

Graph determinant

$$\begin{aligned} \Delta = 1 &+ G_1G_2H_1H_2 + G_2H_2 + G_3G_4H_5 - G_2H4G_4H_3 \\ &+ G_1G_2H_1H_2G_3G_4H_5 + G_2H_2G_3G_4H_5 \end{aligned}$$

$r_1 \to y$ forward paths

$$\begin{aligned} T_1 &= G_1G_2 \\ \Delta_1 &= 1 + G_3G_4H_5 \\ T_2 &= -G_1G_2G_4H_4 \\ \Delta_2 &= 1 \end{aligned}$$

Therefore,

$$\frac{Y(s)}{R_1(s)} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta}$$

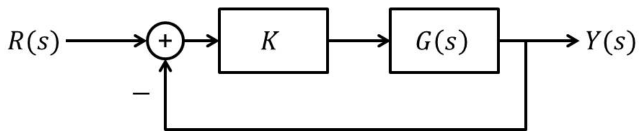
$r_2 \to y$ forward paths

$$\begin{aligned} \bar{T}_1 &= -G_3G_4 \\ \bar{\Delta}_1 &= 1 + G_1G_2H_1H_2 + G_2H_2 \\ \bar{T}_2 &= G_2G_3G_4H_3 \\ \bar{\Delta}_2 &= 1 \end{aligned}$$

Therefore,

$$\frac{Y(s)}{R_2(s)} = \frac{\bar{T}_1\bar{\Delta}_1 + \bar{T}_2\bar{\Delta}_2}{\Delta}$$

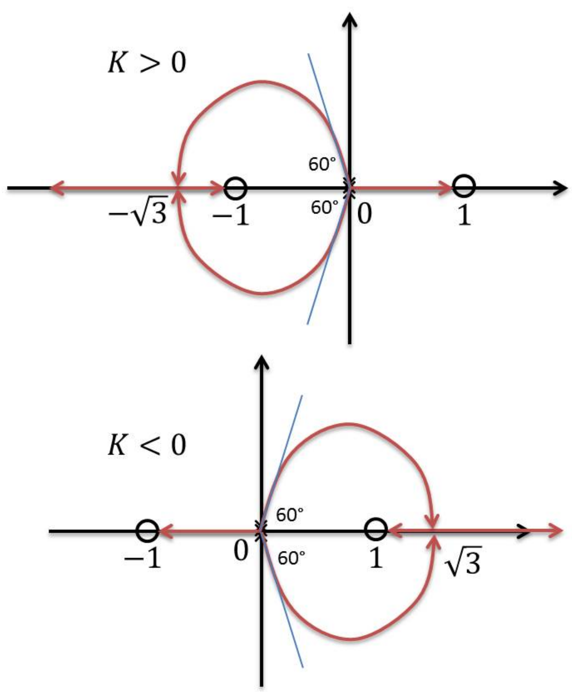
4. Consider the negative unity feedback system shown below:




The plant $G$ is defined as follows:

$$G(s) = \frac{s^2 - 1}{s^3}$$

(a) (5 points) Sketch the root loci for both positive *and* negative $K$.



(b) (5 points) Find the values of $K$ such that the closed-loop system has a pole at 2.

SOLUTION:

$$G(2) = \frac{3}{8} = -\frac{1}{K} \Rightarrow K = -\frac{8}{3}$$

(c) (5 points) Find the values of $K$ such that the closed-loop system is stable.

SOLUTION:

As seen from the root locus plots, there does not exists $K$ such that the system is stable.