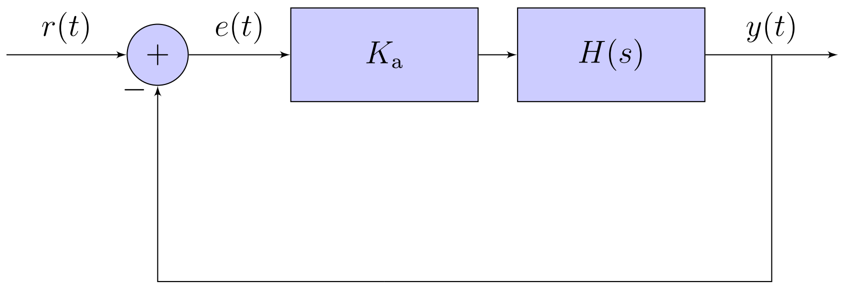
UCLA — Electrical Engineering Dept.
EE141: Principles of Feedback Control
Midterm Exam
Wednesday, February 3, 2016

Solutions

1. A two-phase AC induction motor has transfer function

$$H(s) = \frac{K_\mathrm{m}}{s(T_\mathrm{m}s + 1)},$$

with motor constant $K_\mathrm{m} = 2.5$ and motor time constant $T_\mathrm{m} = 0.1\,\mathrm{s}$. The motor is enclosed in a simple unity feedback loop as in the following diagram:



(a) (1 point) Write down the closed-loop transfer function, $T(s) = \frac{Y(s)}{R(s)}$.

Solution:

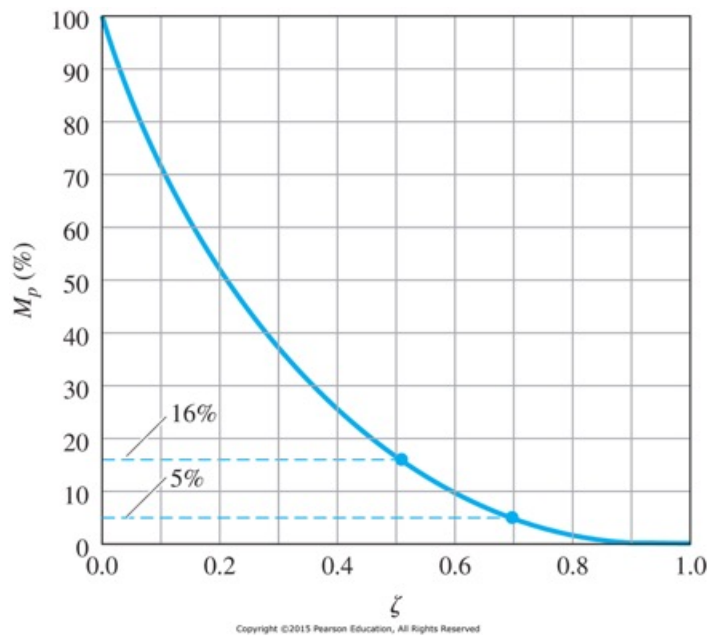
$$T(s) = \frac{K_\mathrm{a}H(s)}{1 + K_\mathrm{a}H(s)} = \frac{K_\mathrm{a}K_\mathrm{m}}{T_\mathrm{m}s^2 + s + K_\mathrm{a}K_\mathrm{m}} = \frac{25K_\mathrm{a}}{s^2 + 10s + 25K_\mathrm{a}}$$

Assume $K_\mathrm{a} = 1$.

(b) (2 points) What are the undamped natural frequency, ω_n, and the damping factor, ζ?

Solution: $\omega_\mathrm{n} = 5\,\mathrm{rad/s}$, $2\zeta\omega_\mathrm{n} = 10 \Rightarrow \zeta = 1$.

(c) (3 points) What are the percent overshoot, M_p, and the settling time, t_s? You may consult the following figure:



Solution: $M_\mathrm{p} = 0\,\%$. Because $\sigma = \zeta\omega_\mathrm{n} = 5$, $t_\mathrm{s} = \frac{4.6}{5} = 0.92\,\mathrm{s}$.

(d) (4 points) What is the expression of the error function, $e(t)$, if the input, $r(t)$, is a unit-step function?

Solution:

$$E(s) = (1 - T(s))\,\frac{1}{s} = \frac{1}{s} - \frac{25}{s(s+5)^2} = \frac{1}{s} - \left(\frac{A_1}{s} + \frac{A_2}{s+5} + \frac{A_3}{(s+5)^2}\right),$$

with $A_1 = 1$, $A_2 = -1$, $A_3 = -5$, therefore

$$e(t) = \mathrm{e}^{-5t} + 5t\mathrm{e}^{-5t}, \quad t > 0.$$

(e) (3 points) When the input, $r(t)$, is a unit-step function, what is the steady-state error, e_ss? Use the final value theorem and check your result with the answer to the previous question. What is the type of this system and is your answer consistent with the system's type?

Solution: By the FVT

$$e_\mathrm{ss} = \lim_{s\to 0} \frac{s^2 + 10s}{(s+5)^2} = 0 = \lim_{t\to\infty} e(t).$$

The system is type 1 and we expect the steady-state error to a step input to be zero.

Now let K_a be a variable. The following answers are independent of the previous questions.

(f) (2 points) Compute the controller gain, K_a that will achieve a damping factor $\zeta = 1/\sqrt{2} \approx 0.7$. What is the undamped natural frequency, ω_n?

Solution: If $\zeta\omega_\mathrm{n} = 5$ and $\zeta = 1/\sqrt{2}$, then $\omega_\mathrm{n} = 5\sqrt{2} \approx 7$. Because $\omega_\mathrm{n} = 5\sqrt{K_\mathrm{a}}$, then $K_\mathrm{a} = 2$.

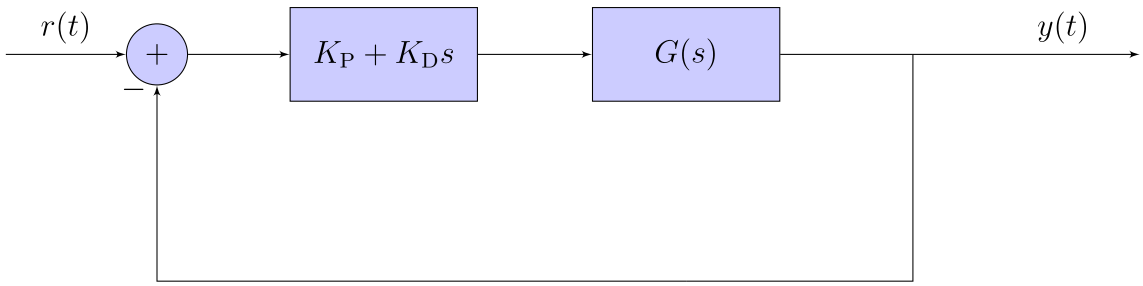
(g) (1 point) What is the range of values of $0 < K_\mathrm{a} < \infty$ that will make the overall system unstable?

Solution: Solving for the roots of the characteristic equation, we have:

$$s^2 + 10s + 25K_\mathrm{a} = 0,$$

which yields $s = -5 \pm 5\sqrt{1 - K_\mathrm{a}}$. No poles will ever be in the RHP.

2. Consider the PD-controlled system in the following figure:



Let $K_{\mathrm{D}} = 1$.

(a) (1 point) Write down the transfer function, $T(s)$, of this system.

Solution:

$$T(s) = \frac{(K_{\mathrm{P}} + s)\, G(s)}{1 + (K_{\mathrm{P}} + s)\, G(s)}.$$

(b) (2 points) Determine the sensitivity of this system's transfer function to K_{P}, i.e., $\mathcal{S}_{K_{\mathrm{P}}}(s) \coloneqq \left(\frac{\mathrm{d}T}{\mathrm{d}K_{\mathrm{P}}}\right)\left(\frac{K_{\mathrm{P}}}{T}\right)$.

Solution:

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}K_{\mathrm{P}}} &= \frac{G(s)\,(1 + (K_{\mathrm{P}} + s)\, G(s)) - G^2(s)\,(K_{\mathrm{P}} + s)}{(1 + (K_{\mathrm{P}} + s)\, G(s))^2} \\ &= \frac{G(s)}{(1 + (K_{\mathrm{P}} + s)\, G(s))^2}.\end{aligned}$$

This yields

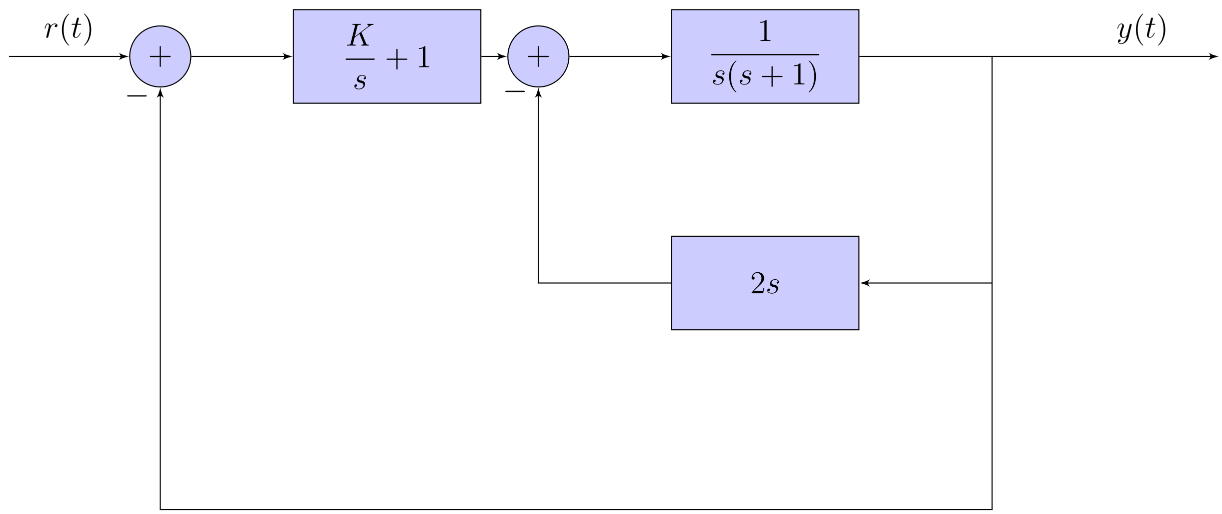
$$\mathcal{S}_{K_{\mathrm{P}}}(s) = \frac{K_{\mathrm{P}}}{(1 + (K_{\mathrm{P}} + s)\, G(s))\,(K_{\mathrm{P}} + s)}$$

(c) (2 points) Write down the expression for the sensitivity at a frequency of 1 rad/s, i.e., compute $|\mathcal{S}_{K_{\mathrm{P}}}(\mathrm{j}\omega)|$ at $\omega = 1$ rad/s, when $G(s) = \frac{1}{s}$.

Solution: At $s = \mathrm{j}$, we have that $G(\mathrm{j}) = \frac{1}{\mathrm{j}} = -\mathrm{j}$ and

$$\begin{aligned}|\mathcal{S}_{K_{\mathrm{P}}}(\mathrm{j})| &= \left|\frac{K_{\mathrm{P}}}{(1 + (K_{\mathrm{P}} + \mathrm{j})G(\mathrm{j}))(K_{\mathrm{P}} + \mathrm{j})}\right| = \left|\frac{K_{\mathrm{P}}}{(2 - \mathrm{j}K_{\mathrm{P}})(K_{\mathrm{P}} + \mathrm{j})}\right| \\ &= \frac{K_{\mathrm{P}}}{\sqrt{K_{\mathrm{P}}^2 + 4}\sqrt{K_{\mathrm{P}}^2 + 1}}\end{aligned}$$

3. Consider the following feedback control system:



(a) (2 points) Write down the transfer function of the system, $T(s)$.

Solution:

$$T(s) = \frac{\frac{s+K}{s}\frac{\frac{1}{s(s+1)}}{1+\frac{2}{s+1}}}{1+\frac{s+K}{s}\frac{\frac{1}{s(s+1)}}{1+\frac{2}{s+1}}} = \frac{\frac{s+K}{s^3+3s^2}}{1+\frac{s+K}{s^3+3s^2}} = \frac{s+K}{s^3+3s^2+s+K}$$

(b) (1 point) Use the Routh method to determine what values of the integral controller gain, $K > 0$, guarantee that the system is stable.

Solution:

$$\begin{array}{c|ll} 3 & 1 & 1 \\ 2 & 3 & K \\ 1 & 1-\frac{K}{3} & 0 \\ 0 & K & \end{array}$$

For stability we need $0 < K < 3$.

(c) (1 point) For the range of K-values that cause the system to be unstable, how many poles reside in the right-hand half-plane?

Solution: When $K > 3$ there are two sign changes in the first column of the Routh table, which means that there are two unstable poles.