Solutions

1. The output of a linear, time-invariant, causal system is $y(t) = \mathrm{e}^{-2t}\sin(t)\mathrm{u}(t)$ when the input is $x(t) = \delta(t) + \mathrm{u}(t)$.

   (a) (5 points) Find the transfer function $H(s)$ of the system and its region of convergence.

   Solution: $Y(s) = 1/[(s+2)^2+1]$, $X(s) = 1 + 1/s$,

   $$H(s) = \frac{Y(s)}{X(s)} = \frac{s}{[(s+2)^2+1](s+1)},$$

   with ROC $= \mathrm{Re}\{s\} > -1$.

   (b) (5 points) Find the frequency response $H(\mathrm{j}\omega)$ of the system. Is $H(\mathrm{j}\omega) = H(s)|_{s=\mathrm{j}\omega}$? Why?

   Solution: The ROC of $H(s)$ includes the imaginary axis, therefore

   $$H(\mathrm{j}\omega) = H(s)\Big|_{s=\mathrm{j}\omega} = \frac{\mathrm{j}\omega}{[(\mathrm{j}\omega+2)^2+1](\mathrm{j}\omega+1)}.$$

2. Consider a linear, time-invariant, causal system described by

$$H(s) = \frac{s}{(s+2)^2}.$$

(a) (5 points) Derive the impulse response function, $h(t)$.

Solution: $H(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2}$; $B = (s+2)^2 H(s)|_{s=-2} = -2$; $0 = H(0) = \frac{A}{2} - \frac{1}{2}$, hence $A = 1$. Therefore, $h(t) = \mathrm{u}(t)\left(\mathrm{e}^{-2t} - 2t\mathrm{e}^{-2t}\right)$.

(b) (10 points) Compute the output, $y(t)$, corresponding to the input $x(t) = \cos(t)$.

Solution: $y(t) = \frac{1}{2}\mathrm{e}^{\mathrm{j}t}H(\mathrm{j}) + \frac{1}{2}\mathrm{e}^{-\mathrm{j}t}H(-\mathrm{j}) = \frac{1}{2}\mathrm{e}^{\mathrm{j}t}\frac{\mathrm{j}}{(\mathrm{j}+2)^2} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}t}\frac{-\mathrm{j}}{(-\mathrm{j}+2)^2}$, therefore

$$y(t) = \mathrm{Re}\left(\mathrm{e}^{\mathrm{j}t}\frac{\mathrm{j}}{(\mathrm{j}+2)^2}\right) = \mathrm{Re}\left(\mathrm{e}^{\mathrm{j}t}\frac{\mathrm{j}(-\mathrm{j}+2)^2}{25}\right) = \mathrm{Re}\left(\mathrm{e}^{\mathrm{j}t}\frac{4+\mathrm{j}3}{25}\right),$$

hence $y(t) = \frac{4}{25}\cos(t) - \frac{3}{25}\sin(t)$.

3. Consider the signal, $x(t)$, whose Fourier transform, $X(\mathrm{j}\omega) = |X(\mathrm{j}\omega)|e^{\mathrm{j}\Theta(\omega)}$, is defined as follows:

$$|X(\mathrm{j}\omega)| = \begin{cases} 1, & -2 < \omega < -1 \text{ and } 1 < \omega < 2 \\ 0, & \text{elsewhere} \end{cases} \qquad \Theta(\omega) = \begin{cases} \frac{\pi}{2}, & \omega < 0 \\ -\frac{\pi}{2}, & \omega > 0 \end{cases}$$

(a) (5 points) Write down the mathematical expression for $X(\mathrm{j}\omega)$ by using the unit step function, $\mathrm{u}(\omega)$.

Solution: $X(\mathrm{j}\omega) = \mathrm{j}\left[\mathrm{u}(\omega+2) - \mathrm{u}(\omega+1) - \mathrm{u}(\omega-1) - \mathrm{u}(\omega-2)\right]$.

(b) (5 points) Do you expect $x(t)$ to be a real signal or an imaginary signal? Why? Answer without computing $x(t)$ explicitly.

Solution: Real, because the amplitude is even and the phase is odd.

(c) (10 points) Derive the expression for $x(t)$. Simplify as much as you can, for example, write all complex exponentials in terms of their real and imaginary parts.

Solution:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathrm{j}\omega)e^{\mathrm{j}\omega t}d\omega = \frac{1}{\pi t}\left[\cos(t) - \cos(2t)\right].$$

4. (a) (8 points) The signal $y(t)$ is obtained by multiplying the signal $x(t)$ of problem 3 with $c(t) = \cos(3t)$, i.e., $y(t) = x(t)\,c(t)$. Compute the Fourier transform of $y(t)$, $Y(\mathrm{j}\omega)$.

> **Solution:**
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> $$Y(\mathrm{j}\omega) = \frac{1}{2\pi} X(\mathrm{j}\omega) * [\pi\delta(\omega+3) + \pi\delta(\omega-3)] = \frac{1}{2} X(\mathrm{j}(\omega+3)) + \frac{1}{2} X(\mathrm{j}(\omega-3)).$$
>
> $$\therefore Y(\mathrm{j}\omega) = \frac{\mathrm{j}}{2}\,[\mathrm{u}(\omega+5) - \mathrm{u}(\omega+4) - \mathrm{u}(\omega+2) + \mathrm{u}(\omega+1) + \mathrm{u}(\omega-1) - \mathrm{u}(\omega-2) - \mathrm{u}(\omega-4) + \mathrm{u}(\omega-5)]\,.$$

(b) (7 points) The signal $y(t)$ is now passed through a linear, time-invariant system with impulse response equal to

$$h(t) = \frac{3}{\pi}\mathrm{sinc}(3t).$$

Compute the output, $z(t)$, of this system. [Hint: It may be easier to work in the frequency domain, i.e., to first compute $Z(\mathrm{j}\omega)$.]

> **Solution:** The system is an ideal low-pass filter with bandwidth = 3 rad/s, therefore $z(t) = -\frac{1}{2}x(t)$.

5. The impulse response of a linear, time-invariant system is

$$h(t) = 2\,\frac{\sin(2t)}{\pi t}\,[\cos(6t)]^2\,.$$

(a) (10 points) Find the frequency response of the system, $H(j\omega)$.

Solution: We can rewrite $h(t) = (2/\pi)\text{sinc}(2t)(1 + \cos(12t))$, which yields

$$H(j\omega) = \frac{1}{2}\,\text{rect}((\omega + 12)/4) + \text{rect}(\omega/4) + \frac{1}{2}\,\text{rect}((\omega - 12)/4).$$

(b) (5 points) Find the output, $y(t)$, when the input is $x(t) = 1 + \sin(2.5t) + \cos(5t)$.

Solution: The signal $x(t)$ has frequency components at frequencies -5,-2.5, 0, 2.5 and 5, but only the component at zero frequency goes through the system, with amplitude given by $H(0) = 1$. The output is $y(t) = 1$.

6. Consider the signal $x(t)$ periodic of period $T = 2$, defined as $x(t) = t^2$, $-1 < t < 1$. The Fourier sine-cosine series expansion of $x(t)$ is given by

$$x(t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi t).$$

(a) (10 points) Using Parseval's theorem, calculate $\sum_{n=1}^{\infty} n^{-4}$.

Solution: $\frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, \mathrm{d}t = \frac{1}{2} \int_{-1}^{1} t^4 \, \mathrm{d}t = 1/5$. Using Parseval's theorem, we have that

$$\frac{1}{5} = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} a_n^2 = \frac{1}{9} + \frac{8}{\pi^4} \sum_{n=1}^{\infty} n^{-4} \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^{-4} = \frac{\pi^4}{8} \left(\frac{1}{5} - \frac{1}{9} \right) = \frac{\pi^4}{90}.$$

(b) (5 points) Write the expression for the mean square error ϵ_1^2 when $x(t)$ is approximated by terms up to the first harmonic.

Solution: $\epsilon_1^2 = \frac{1}{T} \|x(t)\|^2 - a_0^2 - \frac{1}{2} a_1^2 = 1/5 - 1/9 - 8/\pi^4 = 4/45 - 8/\pi^4$.