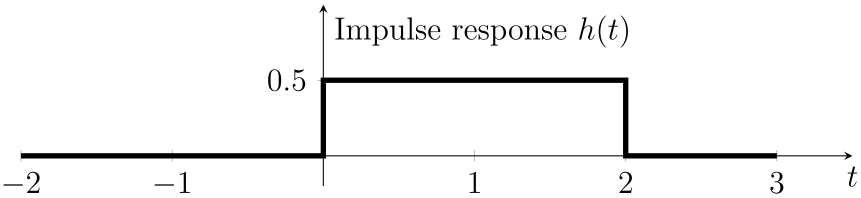
Problem 1 (20 points). Consider the LTI system with impulse response shown below:



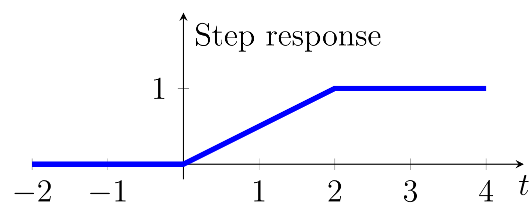
a) (5 points) Determine the step response of the system.

Solution. Step input $\implies$ $u(t) = \mathbf{1}(t)$. Let $y_s(t)$ be the step response.

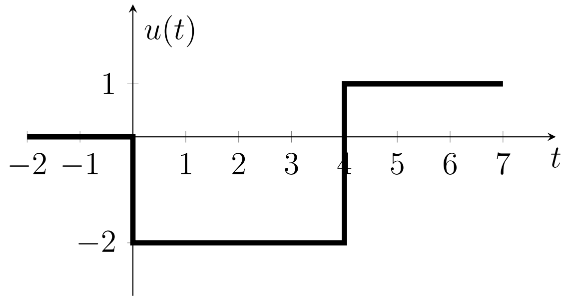
$$y_s(t) = u(t) * h(t) = \int_{-\infty}^{\infty} \mathbf{1}(\tau) h(t-\tau)\, d\tau = \begin{cases} 0 & t < 0 \\ \int_0^t 0.5\, d\tau & 0 \le t < 2 \\ \int_{2-t}^{t} 0.5\, d\tau & t \ge 2 \end{cases}$$

$$= \begin{cases} 0 & t < 0 \\ 0.5t & 0 \le t < 2 \\ 1 & t \ge 2 \end{cases}$$

The step response is plotted below (the plot was not required):



b) (8 points) Consider the input signal $u(t)$ given in the plot below:



Determine the output when the input is $u(t)$. Note that

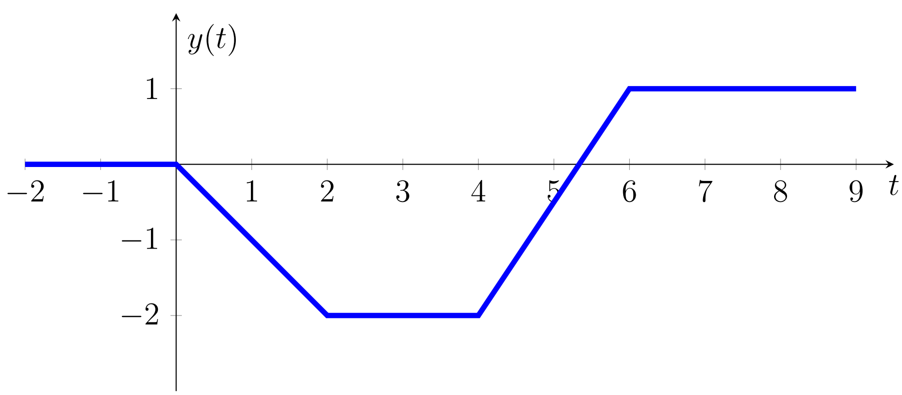
$$u(t) = A\mathbf{1}(t) + B\mathbf{1}(t-4)$$

for appropriate choice of A and B.

Solution. Since $u(t) = -2\mathbf{1}(t) + 3\mathbf{1}(t-4)$, we have $y(t) = -2y_s(t) + 3y_s(t)$:

$$y(t) = \begin{cases} 0 & t < 0 \\ -t & 0 \leq t < 2 \\ -2 & 2 \leq t < 4 \\ \frac{3}{2}t - 8 & 4 \leq t < 6 \\ 1 & t \geq 6 \end{cases}$$

The plot of the output $y(t)$ is below (the plot was not required):

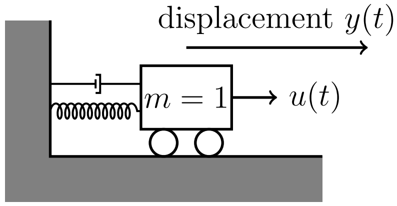


c) (7 points) Find the transfer function $H(s)$ for the system.

Solution.

$$H(s) = \int_0^\infty h(t)e^{-st}\ dt = \frac{1}{2}\int_0^2 e^{-st}\ d\tau = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s}e^{-2s}\right)$$

Problem 2 (35 points). Consider the cart with a spring and damper:



Applied force: $u(t)$
Spring force: $-ky(t)$
Damper force: $-b\dot{y}(t)$
Mass: $m = 1$

a) In parts (i)–(iv) below, assume the input is a step, that is,

$$u(t) = \mathbf{1}(t).$$

Solution: We first write the differential equation governing the displacement as in:

$$u(t) - b\dot{y}(t) - ky(t) = \ddot{y}(t) \tag{1}$$

Assuming that car is at rest initially, we can transform (1) to s-domain as follows:

$$U(s) - bsY(s) - kY(s) = s^2Y(s),$$

which gives us the transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + bs + k}.$$

We first note that to be able to use the standard equations for overshoot, settling time, and rise time we need to have pair of complex conjugate poles. Note that the poles of the characteristic equation of $H(s)$ are $\frac{-b\pm\sqrt{b^2-4k}}{2}$. To have complex poles, for the rest of the questions, we need $b^2 \leq 4k$, that is,

$$\mathbf{b \leq 2\sqrt{k}}.$$

Moreover, the quantities such as rise time, settling time, overshoot does not make sense if our system is not stable, so to ensures stability we also need:

$$\mathbf{b > 0}, \quad \mathbf{k > 0}.$$

Recall the characteristic equation of a standard second-order system is $s^2 + 2\zeta\omega_n s + \omega_n^2$. Therefore, in our case $K = \omega_n^2$, and $b = 2\zeta\omega_n = 2\zeta\sqrt{k}$, which can be rewritten in terms of ζ and ω_n as:

$$\omega_n = \sqrt{K}, \quad \zeta = \frac{b}{2\sqrt{k}}.$$

i) (4 points) Find conditions on b and k to ensure that the rise time t_r satisfies $t_r \leq 0.9$.

We need $t_r = \frac{1.8}{\omega_n} \leq 0.9$ which leads to:

$$\frac{1.8}{\sqrt{k}} \leq 0.9 \implies \sqrt{k} \geq 2 \implies \mathbf{k \geq 4}.$$

Note that to be able to talk about rise time, we need the system to settle in a vicinity of its set point, and hence be stable and note that need $\mathbf{b > 0}$ to ensure stability.

ii) (4 points) Find conditions on b and k to ensure that the percent overshoot M_p satisfies $M_p \leq 5\%$

Solution: To have $M_p \leq 5\%$, we need $\zeta \geq 0.7$. Then this leads to:

$$\frac{b}{2\sqrt{k}} \geq 0.7 \implies \mathbf{b \geq 1.4\sqrt{k}}$$

We also need $\mathbf{k > 0}$ to ensure stability.

iii) (4 points) Find conditions on b and k to ensure that the settling time t_s satisfies $t_s \leq 4$.

Solution: To have $t_s \leq 4$, we need $\frac{4.6}{\zeta\omega_n} \leq 4$. Note that, in our case $2\zeta\omega_n = b$. Then this leads to:

$$\frac{4.6}{\sigma} \leq 4 \implies \frac{4.6}{\frac{b}{2}} \leq 4 \implies \mathbf{b \geq 2.3}.$$

And to ensure stability, we need $\mathbf{k > 0}$.

iv) (3 points) If all three conditions above ($t_r \leq 0.9$, $M_p \leq 5\%$, and $t_s \leq 4$) can be satisfied simultaneously, find a choice of b and k that achieves all three conditions. If b and k cannot be chosen to satisfy all three conditions simultaneously, explain why.

Solution: To be able to satisfy all three conditions simultaneously we need to satisfy:

$$b \geq 2.3, \quad b \geq 1.4\sqrt{k}, \quad k \geq 4, \quad b \leq 2\sqrt{k}. \tag{2}$$

One choice of b and k that satisfies (2) is $k = 9$, $b = 5$.

b) (10 points) Let $b = 6$ and $k = 5$. Determine the output $y(t)$ when the input is the impulse, that is,

$$u(t) = \delta(t).$$

Solution: The transfer function is

$$H(s) = \frac{1}{s^2 + 6s + 5}.$$

Therefore, output is also $Y(s) = H(s)$, since $U(s) = 1$. Then we have:

$$Y(s) = \frac{1}{(s+5)(s+1)}.$$

Performing partial fraction expansion we have:

$$\frac{1}{(s+5)(s+1)} = \frac{C_1}{s+1} + \frac{C_2}{s+5}, \tag{3}$$

and from (3), we get:

$$C_1 = \frac{1}{4}, \text{ and } C_2 = -\frac{1}{4},$$

therefore taking the inverse Laplace transform of $Y(s)$ we get:

$$y(t) = \frac{1}{4}e^{-t} - \frac{1}{4}e^{-5t}, \quad t \geq 0.$$

c) (10 points) Let $b = 0.5$ and $k = 5$. Determine the output $y(t)$ when the input is

$$u(t) = 4\cos(2t) \quad -\infty < t < \infty.$$

Solution: Firstly, note that in this class, we only covered unilateral Laplace transform, because we considered inputs that are defined for $t \geq 0$. So the fact that the input is defined for all times should be a red flag not to use unilateral Laplace transform in this question.

As we discussed in the discussion and in the lecture, the response of a system to a sinusoidal input can be simply determined using the transfer function. Recall that, by using Euler's relation we derived the response of LTI systems to sinusoidal inputs as:

$$u(t) = A\cos(\omega t) \quad \rightarrow \quad y(t) = A|H(jw)|\cos(\omega t + \angle H(jw)). \tag{4}$$

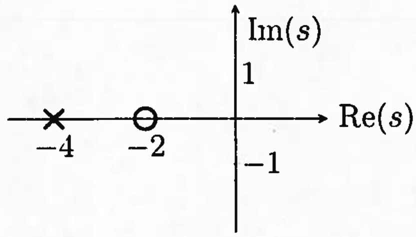
In our case input is $4\cos(2t)$. Hence in (4), $A = 4$ and $\omega = 2$. This means that we need to compute $H(2j)$.

$$H(2j) = \frac{1}{(2j)^2 + 0.5(2j) + 5} = \frac{1}{-4 + j + 5} = \frac{1}{1 + j}.$$

Note that $|H(2j)| = \frac{1}{\sqrt{2}}$, and $\angle H(2j) = -\pi/4$, and therefore the final answer is:

$$4\cos(2t) \quad \rightarrow \quad \frac{4}{\sqrt{2}}\cos\left(2t - \pi/2\right) = 2\sqrt{2}\cos\left(2t - \pi/4\right).$$

Problem 3 (25 points). Consider the causal LTI system described by the pole-zero plot given below:



a) (5 points) The step response has zero steady state error. Determine the system's transfer function $H(s)$.

$$\frac{Y(s)}{R(s)} = H(s) = A \cdot \frac{(s+2)}{(s+4)}$$, $A \rightarrow$ constant

$$E = R - Y = R(s) - R(s)\, A\, \frac{(s+2)}{s+4}$$

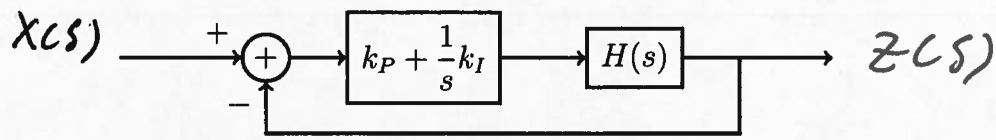
$$= R\left(1 - A\frac{(s+2)}{s+4}\right)$$

$$e_{ss}\,(\text{step}) = \lim_{s\to 0} sE = s\cdot\frac{1}{s}\left(1 - A\frac{(s+2)}{s+4}\right)\left[\lim_{s\to 0}\right]$$

$$= \lim_{s\to 0}\left(1 - \frac{A(s+2)}{s+4}\right) = \lim_{s\to 0}\left[\frac{s+4-As-2A}{s+4}\right]$$

$$\boxed{= \frac{4-2A}{4} = 0 \rightarrow 4 = 2A \rightarrow A = 2}$$

b) (10 points) A proportional-integral controller is used to control the system in unity feedback as shown below:



The closed-loop characteristic equation is

$$7s^2 + 20s + 8 = 0.$$

Determine k_P and k_I.

$$\frac{Z(s)}{X(s)} = \frac{\left[K_P + \frac{1}{s}K_I\right] H(s)}{1 + \left[K_P + \frac{1}{s}K_I\right] H(s)}$$

$$= \frac{\left[K_P + \frac{1}{s}K_I\right] \frac{2(s+2)}{s+4}}{1 + \left[K_P + \frac{1}{s}K_I\right] \frac{2(s+2)}{s+4}} \cdot \frac{s}{s} \cdot \frac{s+4}{s+4}$$

$$= \frac{(sK_P + K_I)\, 2(s+2)}{s(s+4) + [sK_P + K_I]\, 2(s+2)}$$

$$= \frac{2K_P s^2 + s(4K_P + 2K_I) + 4K_I}{s^2(1+2K_P) + s(4K_P + 2K_I + 4) + 4K_I} = \frac{Q(s)}{P(s)}$$

3 b)

$$p(s) = 7s^2 + 20s + 8 \Rightarrow \quad 1 + 2K_p = 7 \quad \longrightarrow \quad K_p = 3$$

$$4K_I = 8 \qquad\qquad K_I = 2$$

$$4K_p + 2K_I + 4 = 20$$

$$4(3) + 2(2) + 4 = 20$$

$$12 + 4 + 4 = 20$$

$$20 = 20 \checkmark$$

c) (5 points) Determine if the closed-loop system is stable.

$7s^2 + 20s + 8 = 0$, $a = 7$, $b = 20$, $c = 8$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow \frac{-20 \pm \sqrt{400 - 4(7)(8)}}{2(7)}$$

$$= \frac{-20 \pm \sqrt{400 - 224}}{14} \quad , \quad \sqrt{400 - 224} = \sqrt{176} \approx 13$$

$$s = \frac{-20 \pm 13}{14}$$

both poles in LHP
so system is stable

d) (5 points) Find the zeros of the closed-loop transfer function.

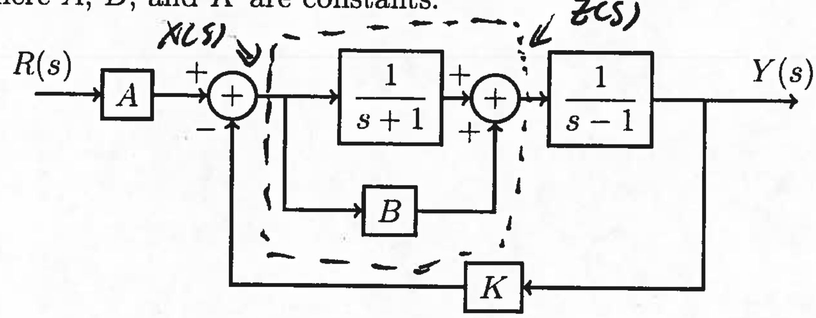
$$Q(s) = 6s^2 + 16s + 8 = 0$$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-16 \pm \sqrt{16^2 - 4(6)(8)}}{2(6)}$$

$$= \frac{-16 \pm \sqrt{256 - 192}}{12} = \frac{-16 \pm \sqrt{64}}{12} = \frac{-16 \pm 8}{12}$$

$$= \frac{-8}{12} \ \& \ \frac{-24}{12} = -\frac{2}{3} \ \& \ -2$$

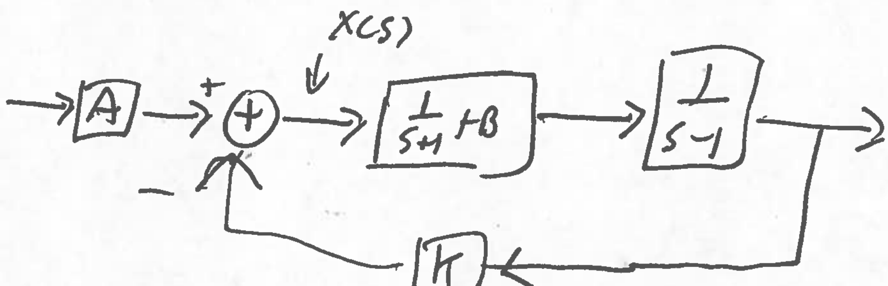
Problem 4 (20 points). Consider the system described by the block diagram below where A, B, and K are constants:



Find constants A, B, and K such that the closed-loop transfer function from $R(s)$ to $Y(s)$ has poles at $s = -3$ and $s = -5$ and the system has zero steady-state error for a step input.

$$Z(s) = \frac{X(s)}{1+s} + X(s) \cdot B$$

$$\frac{Z(z)}{X(s)} = \frac{1}{s+1} + B$$



$$X(s) = RA - YK$$

$$Y(s) = X(s)\left[\frac{1}{s+1} + B\right]\left[\frac{1}{s-1}\right] = [RA - YK]\left[\frac{1}{s+1} + B\right]\left[\frac{1}{s-1}\right]$$

$$= \frac{RA\left[\frac{1}{s+1} + B\right]\left[\frac{1}{s-1}\right]}{1 + K\left[\frac{1}{s+1} + B\right]\left[\frac{1}{s-1}\right]}$$

4) multiply by $\frac{s-1}{s-1}$

$$Y(s) = \frac{RA\left[\frac{1}{s+1} + B\right]}{s-1 + K\left[\frac{1}{s+1} + B\right]} \cdot \frac{(s+1)}{(s+1)}$$

$$= \frac{RA\left[1 + (s+1)B\right]}{(s+1)(s-1) + K\left[1 + (s+1)B\right]}$$

$$= \frac{RA\left[1 + Bs + B\right]}{s^2 - 1 + K\left[1 + Bs + B\right]} = \frac{RA\left[1 + Bs + B\right]}{s^2 - 1 + K + KBs + KB}$$

$$= \frac{Q(s)}{P(s)}, \quad P(s) = (s+3)(s+5) = s^2 + 8s + 15$$

$KB = 8 \qquad KB + K - 1 = 15$

$\downarrow \qquad \downarrow$

$\frac{8}{K} = B \qquad K \cdot \frac{8}{\cancel{K}} + K - 1 = 15 \rightarrow K = 8$

$$\boxed{B = 1 \quad K = 8}$$

Extra workspace (if used, indicate above that extra work appears here).

4)

$$Y(s) = \frac{RA[s+2]}{s^2+8s+15}$$

$$E = R - Y = \frac{1}{s} - \frac{1}{s}\left[\frac{A(s+2)}{s^2+8s+15}\right]$$

$$= \frac{s^2+8s+15 - A(s+2)}{s(s^2+8s+15)}$$

$$e_{ss}\,(\text{step}) = \lim_{s\to 0} sE = \frac{\cancel{s}}{\cancel{s}}\left[\frac{s^2+8s+15 - A(s+2)}{s^2+8s+15}\right]\left(\lim_{s\to 0}\right)$$

$$= \lim_{s\to 0} \rightarrow \frac{15 - A\cdot 2}{15} = 0$$

$$A = \frac{15}{2}$$

Note: Tracking error is defined as the difference between the input & output. Not the input to the summation block.