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EE133A 5/10/2017

Midterm solutions

Problem 1. The *Kronecker product* of two $n \times n$ matrices A and B is the $n^2 \times n^2$ matrix

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{bmatrix}.$$

For example,

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \otimes \begin{bmatrix} 3 & 4 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 9 & 12 \\ -5 & 6 & -15 & 18 \\ 6 & 8 & -3 & -4 \\ -10 & 12 & 5 & -6 \end{bmatrix}.$$

Suppose the $n \times n$ matrices A and B, and an n^2-vector x are given. Describe an efficient method for the matrix-vector multiplication

$$y = (A \otimes B)x = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

(On the right we partitioned x in subvectors x_i of size n.) What is the complexity of your method? How much more efficient is it than a general matrix-vector multiplication of an $n^2 \times n^2$ matrix and an n^2-vector?

Solution. We first compute the n matrix-vector products

$$z_1 = Bx_1, \quad z_2 = Bx_2, \quad \ldots, \quad z_n = Bx_n.$$

This takes $n(2n^2) = 2n^3$ flops. We then compute

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} A_{11}z_1 + A_{12}z_2 + \cdots + A_{1n}z_n \\ \vdots \\ A_{n1}z_1 + A_{n2}z_2 + \cdots + A_{nn}z_n \end{bmatrix},$$

where we partitioned y in subvectors of size n. For each y_k this involves n scalar-vector products $A_{ki}z_i$ and $n-1$ vector additions of size n. This takes $2n^2$ operations, so the total for $y_1, \ldots, y_n$ is $2n^3$. The total flop count is $4n^3$, an order less than for for a general product of this dimension ($2n^4$).

To derive the complexity, we can also note that

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = B \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} A^T,$$

so y can be computed using two matrix-matrix products of size $n \times n$, *i.e.*, in $4n^3$ flops.

Problem 2. Formulate the following problem as a set of linear equations in the form $Ax = b$. Give the numerical values of the elements of the 4×4 matrix A.

Find a polynomial $p(t) = x_1 + x_2 t + x_3 t^2 + x_4 t^3$ that satisfies the four conditions

$$\int_0^1 p(t)dt = b_1, \qquad \int_0^1 tp(t)dt = b_2, \qquad \int_0^1 t^2 p(t)dt = b_3, \qquad \int_0^1 t^3 p(t)dt = b_4.$$

The numbers b_1, b_2, b_3, b_4 on the right-hand sides are given.

Solution. The equations are

$$\begin{aligned}
x_1 \int_0^1 dt + x_2 \int_0^1 tdt + x_3 \int_0^1 t^2 dt + x_4 \int_0^1 t^3 dt &= b_1 \\
x_1 \int_0^1 tdt + x_2 \int_0^1 t^2 dt + x_3 \int_0^1 t^3 dt + x_4 \int_0^1 t^4 dt &= b_2 \\
x_1 \int_0^1 t^2 dt + x_2 \int_0^1 t^3 dt + x_3 \int_0^1 t^4 dt + x_4 \int_0^1 t^5 dt &= b_3 \\
x_1 \int_0^1 t^3 dt + x_2 \int_0^1 t^4 dt + x_3 \int_0^1 t^5 dt + x_4 \int_0^1 t^6 dt &= b_4.
\end{aligned}$$

Evaluating the integrals, we obtain

$$\begin{aligned}
x_1 + x_2/2 + x_3/3 + x_4/4 &= b_1 \\
x_1/2 + x_2/3 + x_3/4 + x_4/5 &= b_2 \\
x_1/3 + x_2/4 + x_3/5 + x_4/6 &= b_3 \\
x_1/4 + x_2/5 + x_3/6 + x_4/7 &= b_4.
\end{aligned}$$

In matrix form, this is

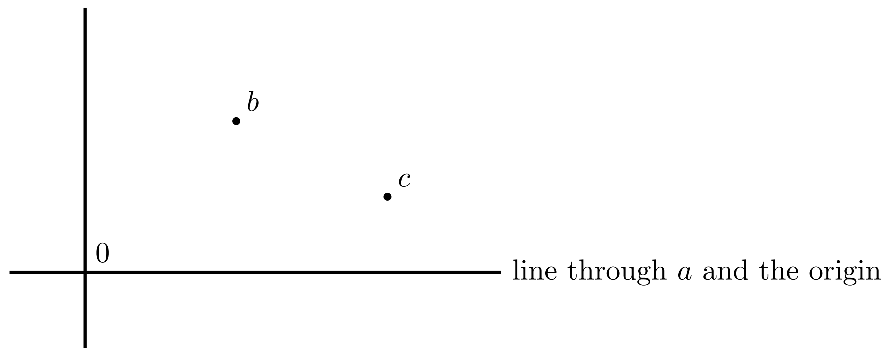
$$\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

Problem 3. Let a be an n-vector with $\|a\| = 1$. Define the $2n \times 2n$ matrix

$$A = \begin{bmatrix} aa^T & I - aa^T \\ I - aa^T & aa^T \end{bmatrix}.$$

1. Show that A is orthogonal.
2. The figure shows an example in two dimensions ($n = 2$). Indicate on the figure the 2-vectors x, y that solve the 4×4 equation

$$\begin{bmatrix} aa^T & I - aa^T \\ I - aa^T & aa^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}.$$



Solution.

1. We verify that $A^TA = I$.

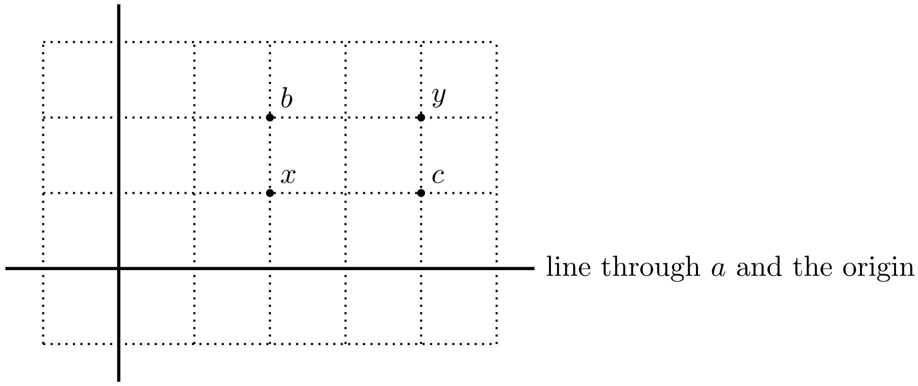
$$\begin{aligned} A^TA &= \begin{bmatrix} aa^T & I - aa^T \\ I - aa^T & aa^T \end{bmatrix} \begin{bmatrix} aa^T & I - aa^T \\ I - aa^T & aa^T \end{bmatrix} \\ &= \begin{bmatrix} aa^Taa^T + I - 2aa^T + aa^Taa^T & 2(aa^T - aa^Taa^T) \\ 2(aa^T - aa^Taa^T) & I - 2aa^T + aa^Taa^T + aa^Taa^T \end{bmatrix} \\ &= \begin{bmatrix} aa^T + I - 2aa^T + aa^T & 2(aa^T - aa^T) \\ 2(aa^T - aa^T) & I - 2aa^T + aa^T + aa^T \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

   The simplifications on line 3 follow because $a^Ta = 1$.

2. Since $A^{-1} = A$, the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} aa^T & I - aa^T \\ I - aa^T & aa^T \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} aa^Tb + (I - aa^T)c \\ (I - aa^T)b + (aa^T)c \end{bmatrix}$$

   The vectors aa^Tb and aa^Tc are the projections of b and c on the line through a on the origin. The vectors $(I - aa^T)b$ and $(I - aa^T)c$ are the projections on the line orthogonal to a.


b
y
x
c
line through a and the origin

Problem 4. Suppose A is an $n \times (n-1)$ matrix with linearly independent columns, and b is an n-vector with $A^T b = 0$ and $\|b\| = 1$.

1. Show that the matrix $\begin{bmatrix} A & b \end{bmatrix}$ is nonsingular with inverse $\begin{bmatrix} A^\dagger \\ b^T \end{bmatrix}$.

2. Let C be any left inverse of A. Show that
$$\begin{bmatrix} C(I - bb^T) \\ b^T \end{bmatrix} \begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}.$$

3. Use the results of parts 1 and 2 to show that $C(I - bb^T) = A^\dagger$.

Solution.

1. We verify that the product of $\begin{bmatrix} A & b \end{bmatrix}$ and the proposed inverse is the identity matrix:
$$\begin{bmatrix} A^\dagger \\ b^T \end{bmatrix} \begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} A^\dagger A & A^\dagger b \\ b^T A & b^T b \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}.$$
We used $A^\dagger A = I$ (by definition), $b^T A = 0$ (by assumption), $A^\dagger b = (A^T A)^{-1} A^T b = 0$ (by definition of pseudo-inverse and $A^T b = 0$), and $b^T b = 1$ (by assumption).

2. We have
$$\begin{bmatrix} C(I - bb^T) \\ b^T \end{bmatrix} \begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} CA - Cbb^T A & C(b - bb^T b) \\ b^T A & b^T b \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}.$$
We used $CA = I$ (by definition of left inverse), $b^T A = 0$, and $b^T b = 1$.

3. Parts 1 and 2 give two expressions for the inverse of $\begin{bmatrix} A & b \end{bmatrix}$. Since the inverse is unique we have
$$\begin{bmatrix} A^\dagger \\ b^T \end{bmatrix} = \begin{bmatrix} C(I - bb^T) \\ b^T \end{bmatrix}.$$

Problem 5. Suppose you have already computed the LU factorization $A = PLU$ of a nonsingular $n \times n$ matrix A. Describe an algorithm for each of the following problems and give its complexity.

1. Compute one particular column of A^{-1}, *i.e.*, compute $(A^{-1})_{1:n,j}$ for one particular index j.
2. Compute the sum of the columns of A^{-1}, *i.e.*, compute $\sum_{j=1}^{n}(A^{-1})_{1:n,j}$.
3. Compute the sum of the rows of A^{-1}, *i.e.*, compute $\sum_{i=1}^{n}(A^{-1})_{i,1:n}$.

Describe each step in your algorithm and give its complexity (number of flops as a function of n). Include in the flop counts terms that are cubic, quadratic, and linear in n. If you know several methods, give the most efficient one (of lowest order, if we exclude the cost of the LU factorization of A).

Solution.

1. To compute $A^{-1}e_j$ we solve the equation $Ax = PLUx = e_j$, using the standard method.
   - $y = P^T e_i$. 0 flops.
   - Solve $Lz = y$ using forward substitution. n^2 flops.
   - Solve $Ux = z$ using back substitution. n^2 flops.

   The total is $2n^2$ flops. Some improvements are possible by using the zero-one structure in the right-hand side of $Lz = y$, but this does not change the order n^2.

2. The sum of the columns is $A^{-1}\mathbf{1}$. To compute this vector we solve $Ax = PLUx = \mathbf{1}$, using the same method. We can skip step one because $P^T\mathbf{1} = \mathbf{1}$.
   - Solve $Lz = \mathbf{1}$ using forward substitution. n^2 flops.
   - Solve $Ux = z$ using back substitution. n^2 flops.

   The total is $2n^2$ flops.

3. The sum of the rows is $\mathbf{1}^T A^{-1}$. To compute this vector we solve $A^T x = U^T L^T P^T x = \mathbf{1}$ as follows.
   - Solve $U^T y = \mathbf{1}$ by forward substitution. n^2 flops.
   - Solve $L^T z = y$ by back substitution. n^2 flops.
   - $x = Pz$. 0 flops.

   The total is $2n^2$ flops.