1

ONE

(a)(i) Yes. $A^TA = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix}$

$$= \begin{vmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \cos\theta\sin\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = I$$

$A^TA = I \Rightarrow$ A has orthonormal columns.

(ii) No. A is tall and so cannot have orthonormal rows.

(iii) No. A does not have orthonormal rows and so is not orthogonal.

(iv) Yes; the left inverse is $A^{-L} = A^T$.

(v) No; A is tall and so cannot have orthonormal rows.

(vi) No; A is not right-invertible and hence not invertible.

(vii) Yes; A matrices have a pseudoinverse. Because A is left invertible, $A^+ = (A^TA)^{-1}A^T = A^T$.

(b) $c = \begin{vmatrix} 0 \\ 0 \\ a \end{vmatrix}$, where $a \neq 0$. E.g. $c = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$

(c) Because B has orthonormal columns (by design) and is square, it is orthogonal. Hence,

$B^TB = BB^T = I \Rightarrow B^{-1} = B^T = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & a \end{vmatrix}$

TWO

(a) True. Say $A, B \in \mathbb{R}^{n \times n}$ are orthogonal, that is $A^TA = AA^T = B^TB = BB^T = I$.

Then if $C = AB$, $C^TC = (AB)^T(AB) = B^T(A^TA)B$

$= B^TB = I$

$CC^T = (AB)(AB)^T = A(BB^T)A^T$

$= AA^T = I$

Hence $C^TC = CC^T = I$ so the product of two orthogonal matrices is orthogonal.

(b) False. $\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = D \Rightarrow D^TD = 0 \neq I$.

D is diagonal but not orthogonal.

(c) True. $A^T = A$, $A^TA = AA^T = I \Rightarrow A^TA = (A)A = A^2 = I$

Hence a symmetric orthogonal matrix A gives $A^2 = I$.

(d) True. $A^T = A$, $A^2 = I \Rightarrow \begin{cases} A^2 = AA = AA^T = I \\ A^2 = AA = A^TA = I \end{cases}$

$\Rightarrow A^TA = AA^T = I$ and so a symmetric matrix with $A^2 = I$ is orthogonal.

THREE

(a) Yes. $f(x) = \frac{x^T a}{\|a\|^2} a = \frac{1}{\|a\|^2}(x^T a)a = \frac{1}{\|a\|^2} a(x^T a) = \frac{1}{\|a\|^2} a(a^T x)$

$= \frac{1}{\|a\|^2}(aa^T)x$

$= Ax$ where $A = \frac{1}{\|a\|^2} aa^T$.

(b) $f(X) = MX$.

$f(\alpha X + \beta Y) = M(\alpha X + \beta Y) = \alpha MX + \beta MY$

$= \alpha f(X) + \beta f(Y)$

$\Rightarrow f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y)$.

f obeys the superposition principle and so is linear.

FOUR

(a) $A = I + UV$; $\hat{A}^{-1} = I - U(I+VU)^{-1}V$

$A\hat{A}^{-1} = (I+UV)(I - U(I+VU)^{-1}V)$

$= I + UV - U(I+VU)^{-1}V - UVU(I+VU)^{-1}V$

$= I + U(I - (I+VU)^{-1} - VU(I+VU)^{-1})V$

$= I + U(I - (I+VU)(I+VU)^{-1})V$

$= I + U(I - I)V = I$

$\Rightarrow A\hat{A}^{-1} = I$ and A is square and so

$\hat{A}^{-1} = A^{-1}$ is the inverse of A.

(b) Direct method:

$A = I + UV \in \mathbb{R}^{m \times m}$.

$A^{-1} \sim O(m^3)$ flops.

(or $O(m^3 + m^2 n)$ if $I+UV$ has to be calculated.

Previous expression:

$B = I + VU \in \mathbb{R}^{n \times n}$; $I + VU \sim O(n^2 m)$ flops

$B^{-1} \sim O(n^3)$ flops

$\Rightarrow A^{-1} = I - U(I+VU)^{-1}V \sim O(n^3 + n^2 m + nm^2)$ flops.

Analysis:

For $m \gg n$, direct method $\sim O(m^3)$ (slow); previous expression $\sim O(m^2)$ (fast). For $m \ll n$, direct method negligible (fast); previous expression $\sim O(n^3)$ (slow).