Midterm Exam Solutions

Note: Please be sure to clearly indicate your answers, and to show how you obtained them. Any plots should be clearly labeled.

1. Fourier Transform (25 points)

(a) (10 points) Prove that when $x(t)$ is odd and real, its Fourier Transform, $X(f)$, is odd and imaginary.

(b) (15 points) Determine the Fourier transform of $\delta(t^2 - a^2)$ where a is a real number.

Solution

(a) The Fourier transform of $x(t)$ can be written as:

$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt = \int_{-\infty}^{+\infty} x(t)\cos(2\pi ft)dt - j\int_{-\infty}^{+\infty} x(t)\sin(2\pi ft)dt$$

We will first show that $X(f)$ is imaginary, and then show that it is odd. Since $x(t)$ and $\cos(2\pi ft)$ are respectively odd and even functions with respect to time, their product $x(t)\cos(2\pi ft)$ would be an odd function in time and therefore the integral over all times would be equal to zero. By contrast, $\sin(2\pi ft)$ is an odd function in time, hence $x(t)\sin(2\pi ft)$ would be an even function of time and have a (in general) non-zero integral over all time. Furthermore, because $x(t)$ and $\sin(2\pi ft)$ are both real, the integral is also real (assuming that the "j" factor is not within the integral, as written above). Therefore, $X(f) = -j\int_{-\infty}^{+\infty} x(t)\sin(2\pi ft)dt$ will be purely imaginary.

For $X(f)$ to be odd, we need to show $X(-f) = -X(f)$. By substituting $(-f)$ in $X(f)$, we have:

$$X(-f) = -j\int_{-\infty}^{+\infty} x(t)\sin(2\pi(-f)t)dt = -j\int_{-\infty}^{+\infty} x(t)(-\sin(2\pi ft))dt = j\int_{-\infty}^{+\infty} x(t)\sin(2\pi ft)dt = -X(f)$$

where $\sin(2\pi(-f)t) = -\sin(2\pi ft)$ because of oddness of $\sin(2\pi ft)$ function. Therefore, $X(f)$ is both imaginary and odd.

(b) From definition of the Fourier transform we have

$$\mathfrak{F}\{\delta(t^2-a^2)\} = \int_{-\infty}^{+\infty}\delta(t^2-a^2)e^{-j2\pi ft}dt = \int_{-\infty}^{+\infty}\delta((t-a)(t+a))e^{-j2\pi ft}dt$$

So, we have two values ($\pm a$)at which the argument of delta function is equal to zero, and where the delta function can be non-zero. At each of these two values, the term related to the other root is fixed, i.e.

$$\delta((t-a)(t+a))\,|_{t=a} = \delta(2a(t-a))$$
$$\delta((t-a)(t+a))\,|_{t=-a} = \delta(-2a(t+a))$$

Therefore from the scaling property of the Fourier transform, applied to the delta function we get $\delta(at) = \frac{1}{|a|}\delta(t)$. Thus:

$$\delta(2a(t-a)) = \frac{1}{2|a|}\delta(t-a)$$
$$\delta(-2a(t+a)) = \frac{1}{2|a|}\delta(t+a)$$

Substituting in the Fourier transform formula above gives:

$$\mathfrak{F}\{\delta(t^2-a^2)\} = \int_{-\infty}^{+\infty}\delta(t^2-a^2)e^{-j2\pi ft}dt = \int_{-\infty}^{+\infty}\frac{1}{2|a|}[\delta(t-a)+\delta(t+a)]e^{-j2\pi ft}dt$$

$$= \frac{1}{2|a|}[e^{-j2\pi fa} + e^{j2\pi fa}] = \frac{\cos(2\pi fa)}{|a|}$$

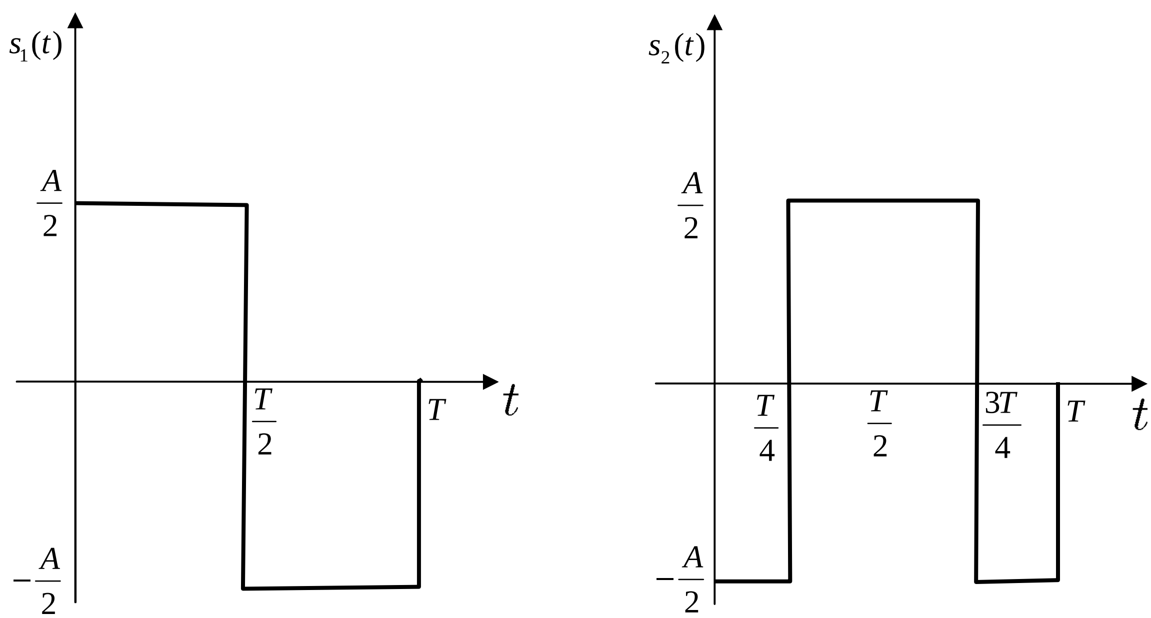
2. Matched (and non-matched) Filter (25 points)

Consider the two pulse shapes $s_1(t)$ and $s_2(t)$ shown below.

(a) (5 points) Plot the impulse response of the matched filters, $h_1(t)$ and $h_2(t)$, matched to the pulses $s_1(t)$ and $s_2(t)$, respectively - in other words, $h_1(t)$ is the filter matched to the pulse $s_1(t)$, and $h_2(t)$ is the filter matched to the pulse $s_2(t)$.

(b) (10 points) Find $y_{11}(T)$, the output that occurs when the pulse $s_1(t)$ is convolved with the matched filter $h_1(t)$ to get $y_{11}(t) = h_1(t) * s_1(t)$, with $y_{11}(t)$ then evaluated at $t = T$.

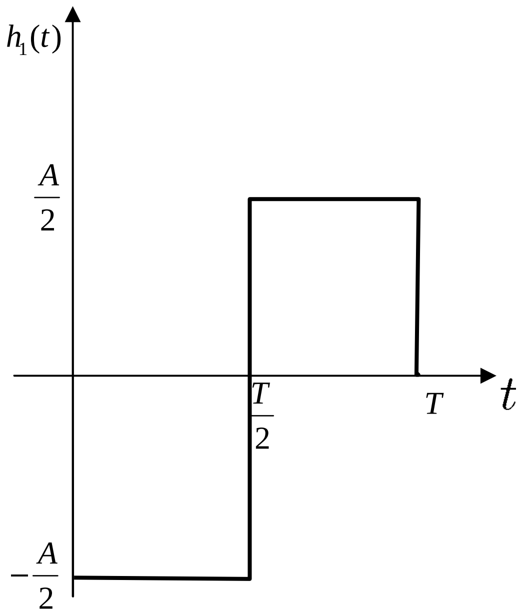
(c) (10 points) Find $y_{21}(T)$, the output that occurs when the pulse $s_1(t)$ is convolved with the $h_2(t)$ to get $y_{21}(t) = h_2(t) * s_1(t)$, with $y_{21}(t)$ then evaluated at $t = T$. (This is not a typo; this problem is asking you to determine the output of a signal with a non-matched filter.)

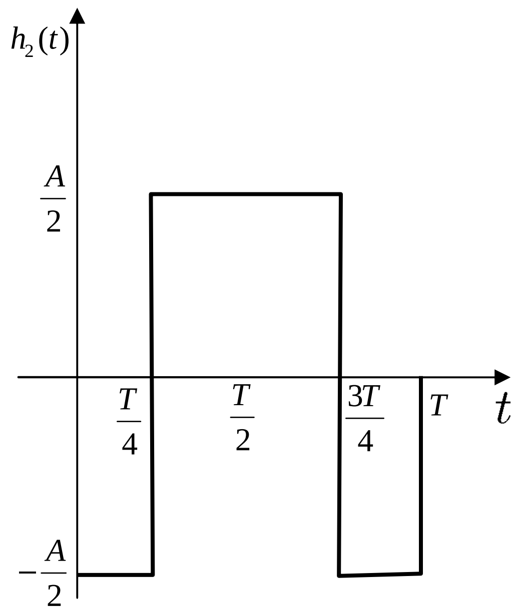


Solution

(a) The matched filter to the $s_1(t)$ and $s_2(t)$ are:

These are shown in the following plots:





(b)

$y_{11}(t) = h_1(t) * s_1(t)$

$0 \le t \le \frac{T}{2}$,

$$y_{11}(t) = \int_0^t -\frac{A}{2}.\frac{A}{2}\,d\tau = -\frac{A^2}{4}t$$

$\frac{T}{2} < t \le T$,

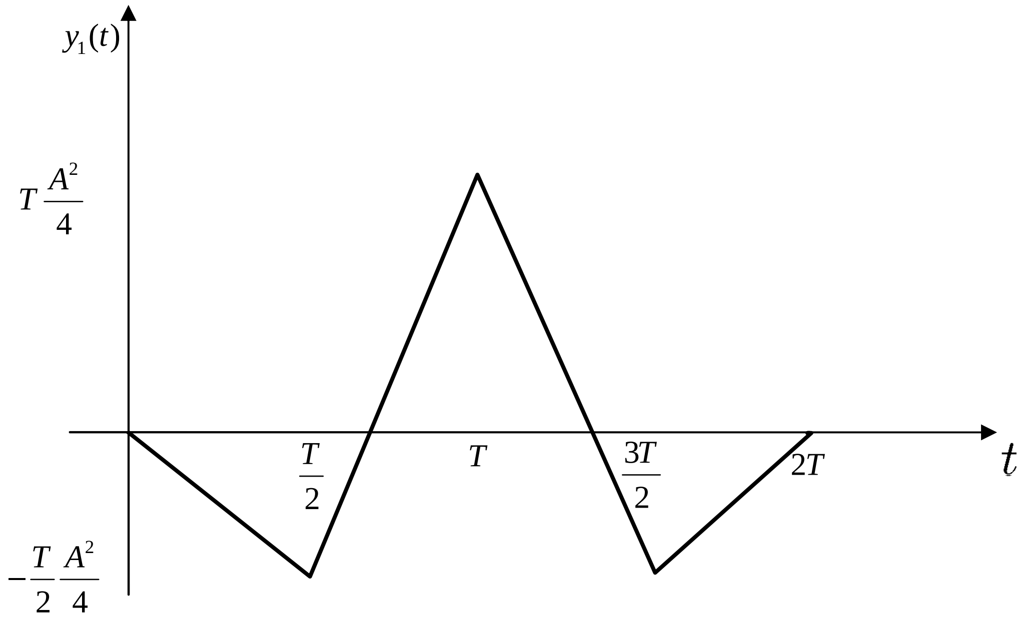
$$y_{11}(t) = \int_0^{t-\frac{T}{2}} \frac{A}{2}.\frac{A}{2}\,d\tau + \int_{t-\frac{T}{2}}^{\frac{T}{2}} -\frac{A}{2}.\frac{A}{2}\,d\tau + \int_{\frac{T}{2}}^{t} -\frac{A}{2}.(-\frac{A}{2})d\tau = \frac{A^2}{4}(3t-2T)$$

$T < t \le \frac{3T}{2}$,

$$y_{11}(t) = \int_{t-T}^{\frac{T}{2}} \frac{A}{2}.\frac{A}{2}\,d\tau + \int_{\frac{T}{2}}^{t-\frac{T}{2}} -\frac{A}{2}.\frac{A}{2}\,d\tau + \int_{t-\frac{T}{2}}^{T} -\frac{A}{2}.(-\frac{A}{2})d\tau = \frac{A^2}{4}(4T-3t)$$

$\frac{3T}{2} < t \le 2T$,

$$y_{11}(t) = \int_{t-T}^{T} -\frac{A}{2}.\frac{A}{2}\,d\tau = -\frac{A^2}{4}(2T-t)$$



The value at T is $\frac{A^2T}{4}$.

(c) $y_{21}(t) = h_2(t) * s_1(t)$, This gives:

$0 \le t \le \frac{T}{4}$,
$$y_{21}(t) = -\frac{A^2}{4}t$$

$\frac{T}{4} < t \le \frac{T}{2}$,
$$y_{21}(t) = \frac{A^2}{4}(t - \frac{T}{2})$$

$\frac{T}{2} < t \le \frac{3T}{4}$,
$$y_{21}(t) = \frac{A^2}{4}(3t - \frac{3T}{2})$$

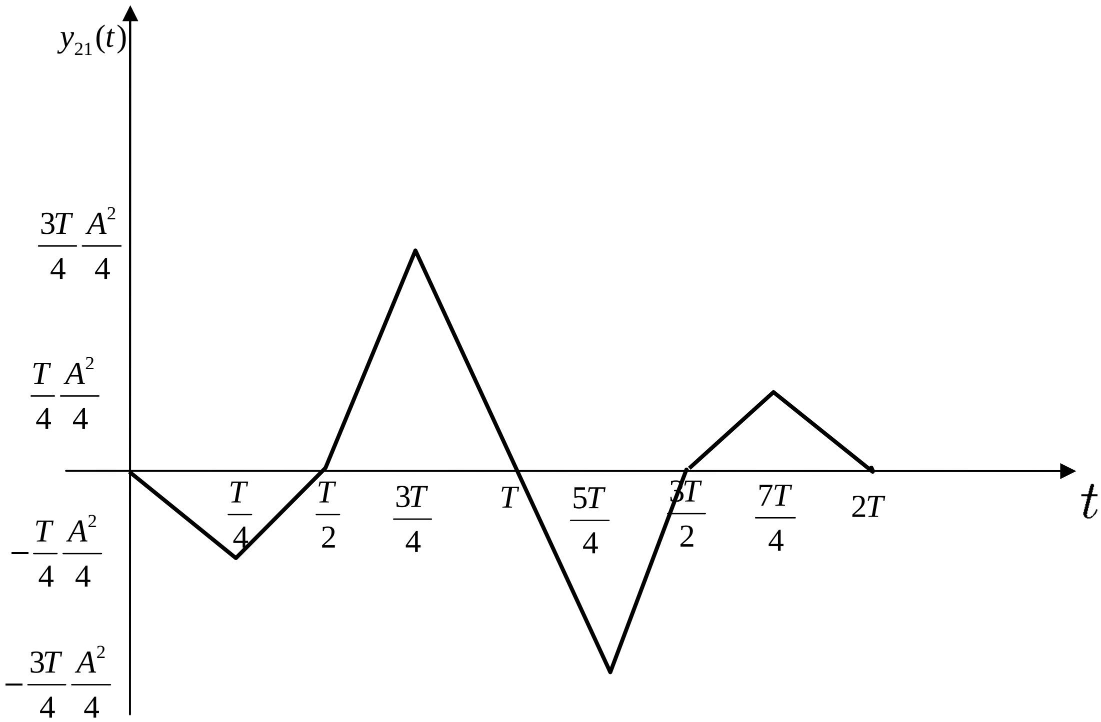
$\frac{3T}{4} < t \le T$,
$$y_{21}(t) = \frac{A^2}{4}(3T - 3t)$$

$T < t \le \frac{5T}{4}$,
$$y_{21}(t) = \frac{A^2}{4}(3T - 3t)$$

$\frac{5T}{4} < t \le \frac{6T}{4}$,
$$y_{21}(t) = \frac{A^2}{4}(3t - \frac{9T}{2})$$

$\frac{6T}{4} < t \le \frac{7T}{4}$,
$$y_{21}(t) = \frac{A^2}{4}(t - \frac{3T}{2})$$

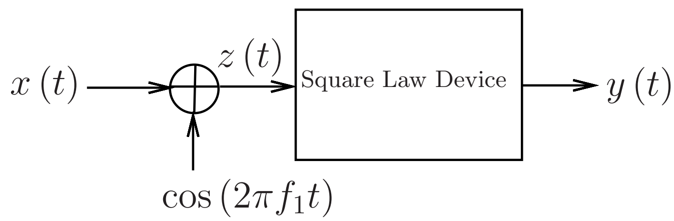
$\frac{7T}{4} < t \le 2T$,
$$y_{21}(t) == \frac{A^2}{4}(2T - t)$$



The value at T is 0.

3. Modulation and Squaring (25 points)

Consider a system shown in the block diagram below.



The deterministic input signal $x(t)$ has a spectrum $X(f)$, which is bandlimited to $|f| < W$ Hz.

a) (15 points) Determine a maximally simplified expression for the spectrum, $Y(f)$, the Fourier transform of the output $y(t)$. Your answer should be a function of $X(f)$.

b) (10 points) Provide a block diagram of a system that can be used to reconstruct $x(t)$ from $y(t)$, being sure to specify any constraints relating f_1 and W that must hold in order for this system to work.

Solution

a)

Let $z(t) = x(t) + \cos(2\pi f_1 t)$

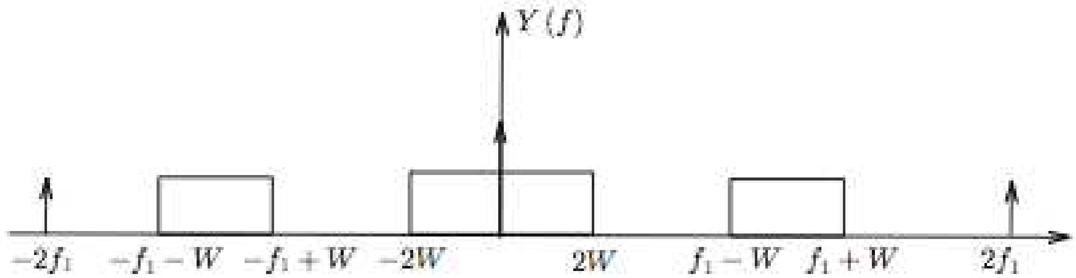
$y(t) = z(t)^2 = x(t)^2 + 2x(t)\cos(2\pi f_1 t) + \cos^2(2\pi f_1 t)$

Applying the trig. identity $\cos^2(A) = \frac{1}{2} + \frac{1}{2}\cos(2A)$,

we have $y(t) = x(t)^2 + 2x(t)\cos(2\pi f_1 t) + \frac{1}{2} + \frac{1}{2}\cos(2\pi 2 f_1 t)$. (Note that even if you don't know this trig identity, you can easily obtain it by associating squaring a cosine in the time domain with the self-convolution of a pair of delta functions in the transform domain.) Taking the Fourier Transform on both sides yields,

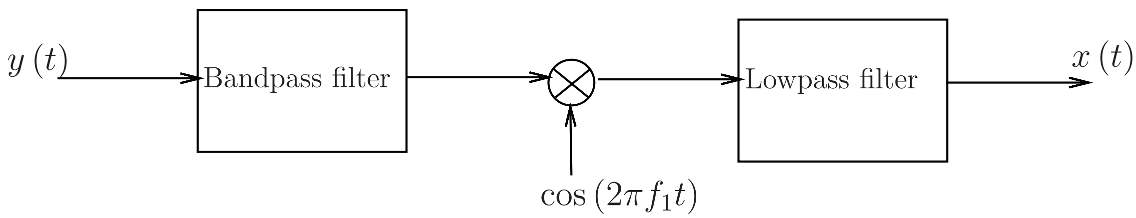
$$Y(f) = X(f) * X(f) + X(f - f_1) + X(f + f_1) + \frac{1}{2}\delta(f) + \frac{1}{4}\left(\delta(f - 2f_1) + \delta(f + 2f_1)\right)$$

In the frequency domain this can be illustrated as follows (where the rectangles below illustrate the locations of frequencies that are occupied, not necessarily the shape of the actual spectrum). Also, plot shows the case where $f_1 > 3W$. If this is not met, then there will be some overlap between components of the plot.



b)

It is possible to reconstruct $x(t)$ perfectly from $y(t)$ using the block diagram shown below.



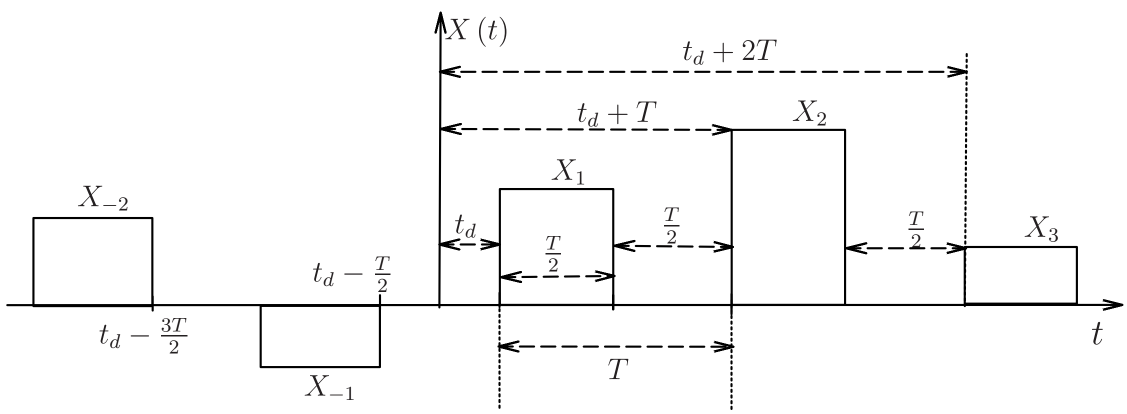
Bandpass filter characteristic: $\begin{cases} 1, & f_1 - W < |f| < f_1 + W \\ 0, & \text{otherwise} \end{cases}$

Lowpass filter characteristic: $\begin{cases} 1, & |f| < W \\ 0, & \text{otherwise} \end{cases}$

In order for this system to work, we need
$f_1 - W > 2W$
Therefore, $f_1 > 3W$

4. Random Processes (25 points)

Consider a sequence of pulses where each pulse lasts T seconds, and consists of an "active region" of duration $T/2$ and a "dead region" of duration $T/2$. The amplitude of each pulse in the active region is X , where X is a random variable with mean μ_x and second moment s . The amplitude in the dead region is zero. Amplitudes in different pulses are independent, identically distributed (i.i.d.), drawn from the random variable X given above.



The active regions start at times $nT + t_d$. The random variable t_d is uniformly distributed over $[0,T)$. The process $X(t)$ is stationary and ergodic in the autocorrelation.

(a) (12 points) Find $R_X(T)$, the autocorrelation corresponding to $R_X(\tau)$ evaluated at $\tau = T$.

(b) (13 points) Find $R_X\left(\frac{3T}{2}\right)$, the autocorrelation corresponding to $R_X(\tau)$ evaluated at $\tau = \frac{3T}{2}$.

Solution

a) We consider two cases:

- Case A: $X(0)$ and $X(T)$ are in different active regions. For this case $E\left[X(0)X(T) \mid A\right] = \mu_x^2$, since the expectation of the product is the product of the expectations.

$X(0)$ and $X(T)$ can only be in different active regions when t_d is in the second half of the $[0,T)$ interval. (When it is in the first half of the $[0,T)$ interval, then both 0 and T will be in dead regions.) The probability that t_d is in the second half of the $[0,T)$ interval is 0.5.

- Case B: Either or both of $X(0)$ and $X(T)$ are in a dead region(s). For this case $E\left[X(0)X(T)\,|\,B\right]=0$

Note that there is no possibility for $X(0)$ and $X(T)$ to be in the same active region.

In order to calculate $R_X(T)$, we use:

$$R_X(T)=\sum_i E\left[X(0)X(T)\,|\,i\right]P(i)$$

where i denotes the cases 1 and 2. This gives $R_X(T)=\frac{1}{2}\mu_x^2$.

b) 0 and $\frac{3T}{2}$ cannot be in the same active region. Additionally, if 0 is in an active region, then T must also be in an active region, which means $\frac{3T}{2}$ is in a dead region (since active regions have duration T/2). Similarly, if 0 is in a dead region, $\frac{3T}{2}$ is in an active region. Thus, there is no possibility for 0 and $\frac{3T}{2}$ to both be in different active regions. Thus, $R_X\left(\frac{3T}{2}\right)=0$.