Student Name: ___________________________

Student ID: ___________________________

Name of the student sitting to your left: ______________

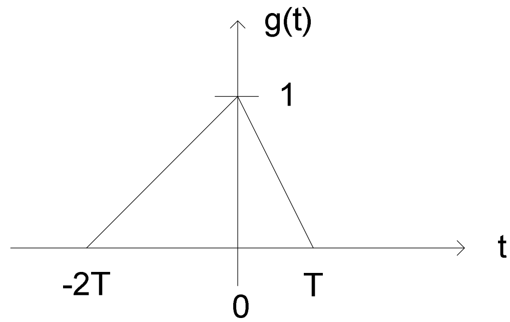
Name of the student sitting to your right: ____________

QUESTION	SCORE	FULL SCORE
1		20
2		20
3		20
4		20
5		20
TOTAL		100

Note: Please be sure to clearly indicate your answers, and to show how you obtained them.

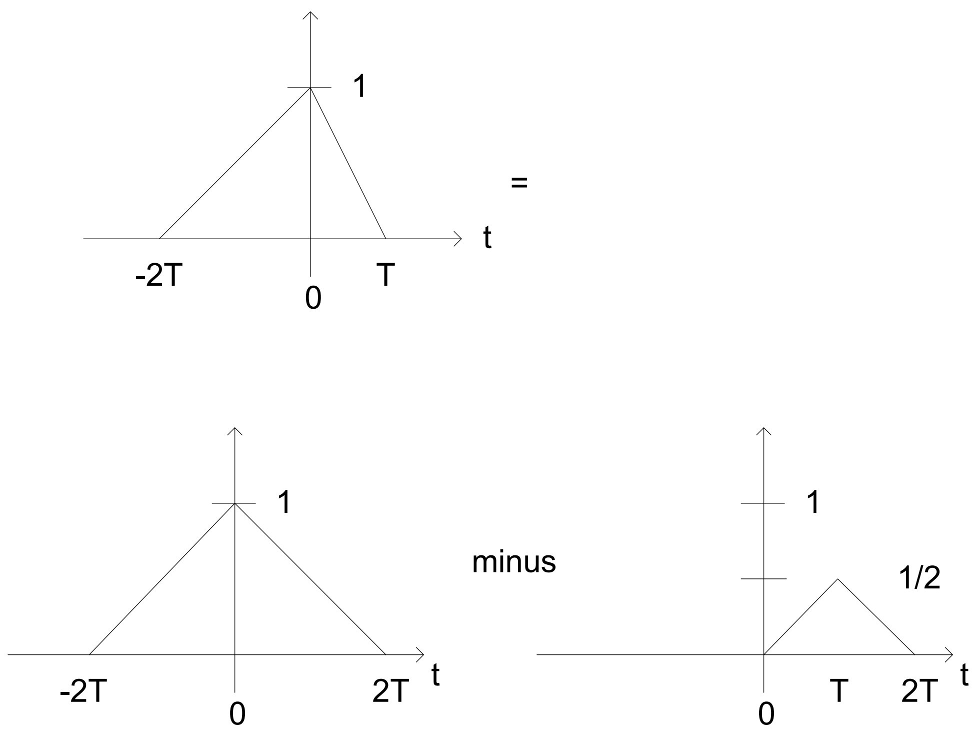
1. (20 points) Consider the $g(t)$ shown below. Assume that

$g(t) = 0$ when $t > T$, $t < -2T$. Compute the Fourier transform of $g(t)$.



Solution to problem 1:

$g(t)$ can be decomposed into two triangle functions as shown below



So $g(t)$ can be written as follows

$g(t) = \text{tri}\left(\frac{t}{2T}\right) - \frac{1}{2}\text{tri}\left(\frac{t-T}{T}\right)$, where tri($t$/T) denotes a function with an isosceles triangle shape and having height 1 at its peak and width 2T at its base.

Using Fourier transform pairs,

$\text{tri}\left(\frac{t}{T}\right) \Leftrightarrow T\text{sinc}^2(Tf)$ (even if you didn't explicitly know this transform pair, it is easy to derive since you know that a rect function transforms to a sinc function). Then,

$$\delta(t-T) \Leftrightarrow \exp(-j2\pi Tf)$$

gives $G(f) = 2T\text{sinc}^2(2Tf) - \frac{1}{2}T\text{sinc}^2(Tf)\exp(-j2\pi Tf)$

Alternative methods are possible to solve this problem, resulting in different but equivalent forms of the above solution.

2 . (20 points) Consider a system in which a DSB-SC AM signal is generated by multiplying the message signal *m(t)* with a periodic rectangular waveform *s(t)* as shown in Fig. 1 and filtering the product with a unit gain band-pass filter (BPF) centered at $1/T_p$, where T_p is the period of the rectangular waveform. Also, the BPF has a bandwidth of $2W$, where W is the bandwidth of the message signal. You can assume that the bandwidth of the message signal is much smaller than $1/T_p$.

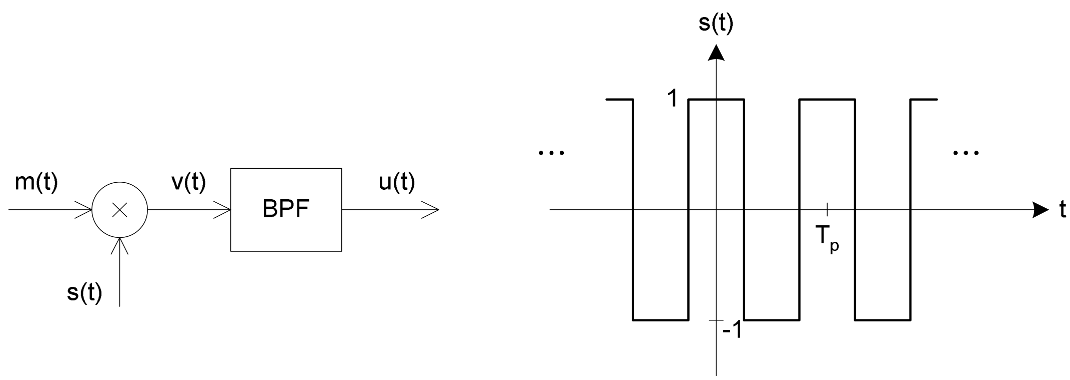


Fig. 1: Block diagram of a DSB-SC modulator using rectangular waveform

(a) (14 points) Find *V(f)*, the Fourier transform of *v(t)*, which is the signal before passage through the BPF. Your expression for *V(f)* can be expressed as a function of *M(f)*, where *M(f)* is the Fourier transform of the input signal *m(t).*

(b) (6 points) Give the correctly scaled time domain expression for *u(t)*, expressed as a function of the input signal *m(t)*.

Solution to problem 2:

(a) Since $s(t)=\left[2\,\text{rect}\left(\frac{t}{T_p/2}\right)*\sum_{n=-\infty}^{\infty}\delta(t-nT_p)\right]-1$

$$\Rightarrow S(f)=\left[2\frac{T_p}{2}\text{sinc}\left(f\frac{T_p}{2}\right)\frac{1}{T_p}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_p}\right)\right]-\delta(f)$$

$$=\left[\sum_{n=-\infty}^{\infty}\text{sinc}\left(\frac{n}{2}\right)\delta\left(f-\frac{n}{T_p}\right)\right]-\delta(f)$$

Let $v(t)=m(t)s(t)$,

$$\Rightarrow V(f)=M(f)*S(f)=\left[\sum_{n=-\infty}^{\infty}\text{sinc}\left(\frac{n}{2}\right)M\left(f-\frac{n}{T_p}\right)\right]-M(f)$$

(b) The bandpass filter will cut off all frequencies except the ones centered at $\frac{1}{T_p}$,

that is for $n=\pm 1$. Thus, the output spectrum is

$$U(f)=\text{sinc}\left(\frac{1}{2}\right)M\left(f-\frac{1}{T_p}\right)+\text{sinc}\left(-\frac{1}{2}\right)M\left(f+\frac{1}{T_p}\right)$$

$$=\frac{2}{\pi}M\left(f-\frac{1}{T_p}\right)+\frac{2}{\pi}M\left(f+\frac{1}{T_p}\right)$$

$$=\frac{4}{\pi}M(f)*\frac{1}{2}\left[\delta\left(f-\frac{1}{T_p}\right)+\delta\left(f+\frac{1}{T_p}\right)\right]$$

Taking the inverse Fourier transform of the previous expression, we obtain

$$u(t)=\frac{4}{\pi}m(t)\cos\left(2\pi\frac{1}{T_p}t\right)$$

which has the form of a DSB-SC AM signal, with $c(t)=\frac{4}{\pi}\cos\left(2\pi\frac{1}{T_p}t\right)$.

3. (20 points) The Fourier transform *M(f)* of a message signal *m(t)* is given in Fig. 2. Note that in contrast with the usual case in which message signals are purely real which would lead to a Hermitian Fourier transform (where Hermitian means that the real part is even and the imaginary part is odd), in this case *M(f)* is purely real but not even, meaning the *m(t)* must be complex. This message signal *m(t)* is modulated that results in a modulated signal $x_c(t)$, which has Fourier transform $X_c(f)$ as shown in Fig. 3.

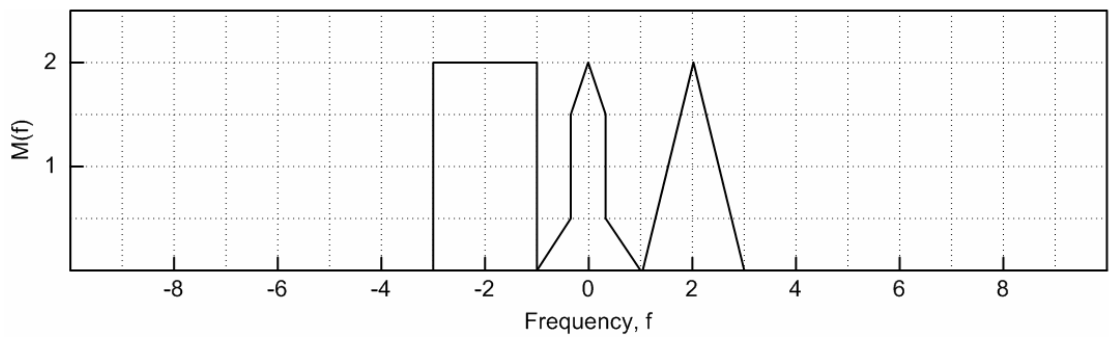


Fig. 2: The Fourier transform *M*(*f*) of the message signal *m*(*t*)

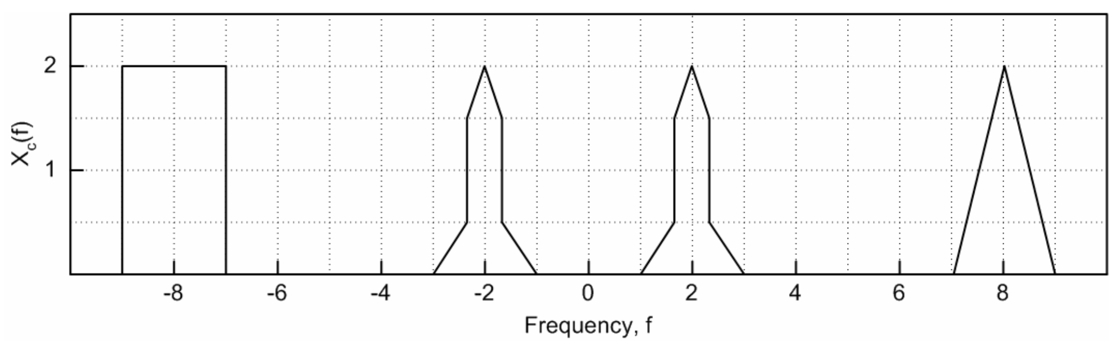
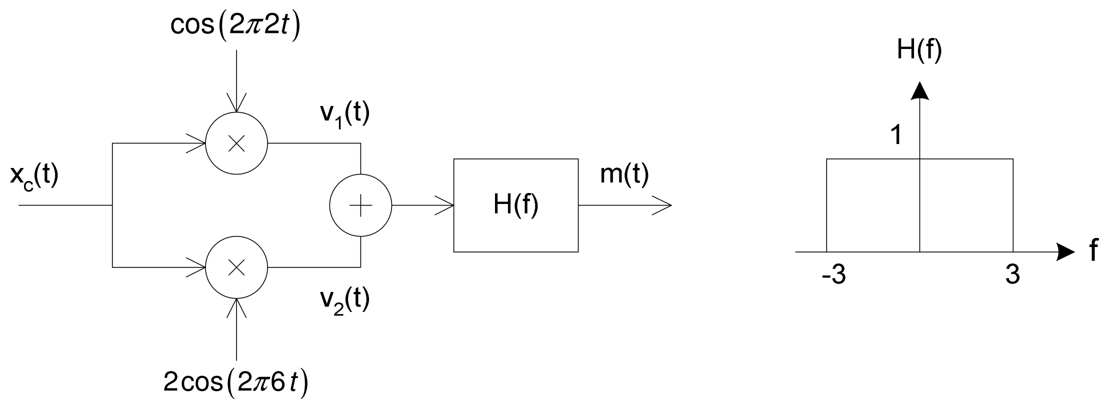


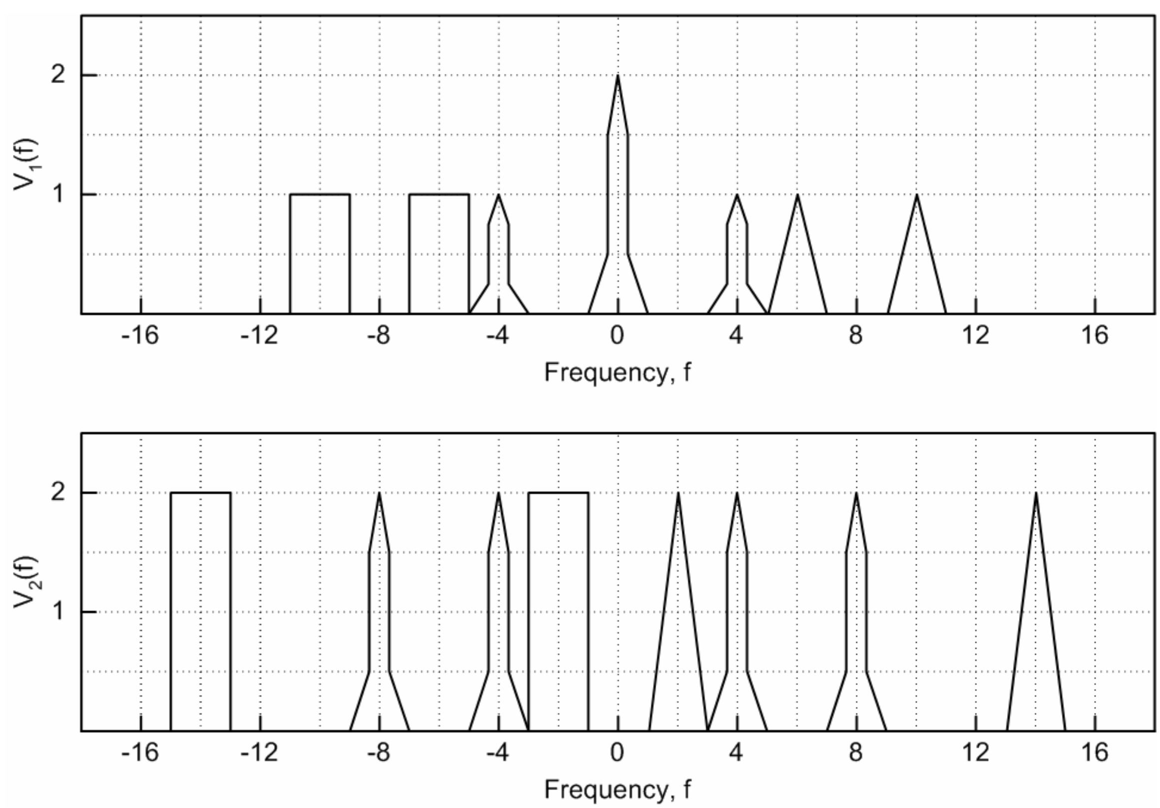
Fig. 3: The Fourier transform $X_c(f)$ of the modulated signal $x_c(t)$

Design a receiver that will successfully recover *m(t)* from $x_c(t)$. Your answer should be in the form of a block diagram. The receiver must only consist of blocks that multiply by appropriate (possibly scaled) sinusoids, summation blocks and low-pass filters. Your receiver can NOT include any complex functions; all functions used in the receiver must be purely real. In order to get full credit, you must explicitly state the attributes of any sinusoidal functions that you use, and the bandwidth and gain of any low-pass filters that you use.

Solution to problem 3:



This can be seen visually by looking at the Fourier transforms of $v_1(t)$ and $v_2(t)$, where $v_1(t) = x_c(t)\cos(2\pi 2t)$ and $v_2(t) = x_c(t)2\cos(2\pi 6t)$. Once these two signals are summed and pass through the low-pass filter with the transfer function *H(f)* specified above, you will be able to reconstruct *m(t)*.



4. (20 points) Consider a random process *Y(t)* defined by

$$Y(t) = \int_0^t X(\tau)\,d\tau$$

and $X(t)$ is given by

$$X(t) = A\cos(\omega t) + B\sin(\omega t),$$

where ω is constant and A and B are Gaussian random variables with mean μ and variance σ^2.

(a) (10 points) What limitation on the random variables A and B would make *X(t)* wide sense stationary?

(b) (10 points) For this part of the problem, let A and B be independent Gaussian random variables with zero mean and unit variance. Determine mean and the variance of *Y(t)* at $t = t_k$.

Note: Some potentially useful trigonometry identities.

$$\cos(x)\cos(y) = \frac{\cos(x-y) + \cos(x+y)}{2}$$
$$\sin(x)\sin(y) = \frac{\cos(x-y) - \cos(x+y)}{2}$$
$$\sin(x)\cos(y) = \frac{\sin(x+y) + \sin(x-y)}{2}$$
$$\cos(x)\sin(y) = \frac{\sin(x+y) - \sin(x-y)}{2}$$

Solution to problem 4:

(a) For *X(t)* to be WSS, it must be stationary in the mean and in the autocorrelation. From

$$E\left[X(t)\right] = E[A]\cos(\omega t) + E[B]\sin(\omega t),$$

If $E[A] = E[B] = 0$, then the mean will be 0 for all t, and thus *X(t)* will be stationary in the mean.

The autocorrelation function expression is

$$\begin{aligned} R_X(t_1,t_2) &= E\left[X(t_1)X(t_2)\right] \\ &= E\left[\left(A\cos(\omega t_1) + B\sin(\omega t_1)\right)\left(A\cos(\omega t_2) + B\sin(\omega t_2)\right)\right] \\ &= E\left[A^2\right]\cos(\omega t_1)\cos(\omega t_2) + E[AB]\cos(\omega t_2)\sin(\omega t_1) \\ &\quad + E[AB]\cos(\omega t_1)\sin(\omega t_2) + E\left[B^2\right]\sin(\omega t_1)\sin(\omega t_2) \\ &= \frac{1}{2}E\left[A^2\right]\cos\left(\omega(t_2 - t_1)\right) + \frac{1}{2}E\left[B^2\right]\cos\left(\omega(t_2 - t_1)\right) + E[AB]\sin\left(\omega(t_2 + t_1)\right) \\ &= \frac{1}{2}E\left[A^2 + B^2\right]\cos(\omega\tau) + E[AB]\sin\left(\omega(t_2 + t_1)\right) \text{ where } \tau = t_2 - t_1 \end{aligned}$$

In order for this result to depend only on τ, we must have $E[AB] = 0$.

(b) Since A and B are independent Gaussian random variables with zero mean and unit variance, we know that $E[A] = E[B] = E[AB]= 0$ and $E[A^2]= E[B^2]=1$.

$$\begin{aligned} Y(t_k) &= \int_0^{t_k} A\cos(\omega\tau) + B\sin(\omega\tau)\,d\tau \\ &= \left[A\frac{\sin(\omega t)}{\omega} - B\frac{\cos(\omega t)}{\omega}\right]_0^{t_k} \\ &= A\frac{\sin(\omega t_k)}{\omega} - B\frac{\cos(\omega t_k) - 1}{\omega} \end{aligned}$$

$$E\left[Y(t_k)\right] = E[A]\frac{\sin(\omega t_k)}{\omega} - E[B]\frac{\cos(\omega t_k) - 1}{\omega} = 0$$

$$Var\left[Y\left(t_k\right)\right]=E\left[Y^2\left(t_k\right)\right]-E\left[Y\left(t_k\right)\right]^2$$

$$=E\left[A^2\right]\left(\frac{\sin\left(\omega t_k\right)}{\omega}\right)^2+E\left[B^2\right]\left(\frac{\cos\left(\omega t_k\right)-1}{\omega}\right)^2-2E\left[AB\right]\frac{\sin\left(\omega t_k\right)}{\omega}\frac{\cos\left(\omega t_k\right)-1}{\omega}$$

$$=\frac{1}{\omega^2}\left(\sin^2\left(\omega t_k\right)+\cos^2\left(\omega t_k\right)-2\cos\left(\omega t_k\right)+1\right)=\frac{2}{\omega^2}\left(1-\cos\left(\omega t_k\right)\right)$$

Thus, $Y(t_k)$ has zero mean and variance $\sigma_Y^2=\frac{2}{\omega^2}\left(1-\cos\left(\omega t_k\right)\right)$.

5. Consider a system like the one discussed in lecture containing a transmit filter such that the transmitted pulses have the form *g(t)*, which is nonzero only in the interval $(0,T)$. Assume for this problem that the channel has infinite bandwidth, so that the only impairments arising from passage through the channel are due to the addition of noise. Normally we have assumed an additive white Gaussian noise model, in which the noise introduced in the channel has a flat power spectral density $S_n(f) = N_o/2$. In this problem, however, we consider a system in which the noise added is colored Gaussian noise $n_c(t)$, which is modeled by passing additive white Gaussian noise *n(t)* through the filter *C(f)* as shown in Figure 5. Thus, under this scenario the signal at the input to the matched filter H(f) is $g(t)+n_c(t)$, assuming a "one-shot" transmission where we transmit a single pulse. The output of the matched filter is sampled at time *T*, yielding a signal *y(T)* which is the sum of contributions due to the signal and noise components respectively; i.e. $y(T) = y_s(T) + y_n(T)$.

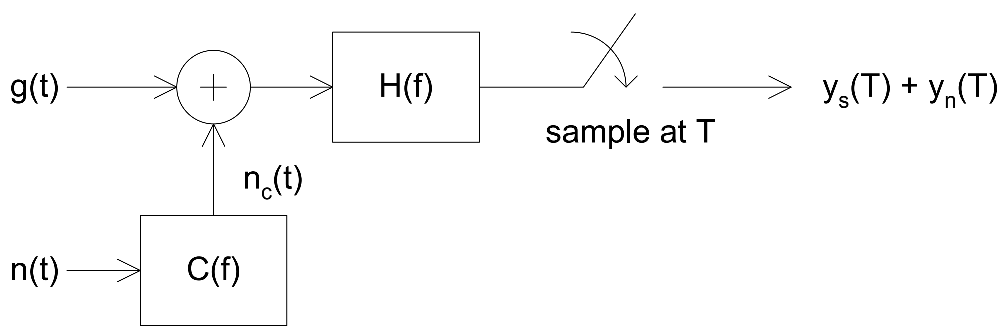


Fig. 5: Matched filter for colored Gaussian noise

Design a filter *H(f)* which maximizes the SNR, which is the ratio of the signal power to the noise power, measured at the sampled matched filter output. Show the maximum value of SNR.

<u>Solution to problem 5:</u>

When *C(f)* = 1 (white noise case), the noise power at the output of filter *h(t)* due to *n(t)* is

$$P_N = \frac{N_0}{2}\int_{-\infty}^{\infty}\left|H(f)\right|^2 df \ .$$

For *C(f)* ≠ 1, the noise power becomes

$$P_{N_c} = \frac{N_0}{2}\int_{-\infty}^{\infty}\left|C(f)H(f)\right|^2 df$$

The signal power at the sampled matched filter output is still the same as the white noise case.

$$P_S = \left|\int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT}df\right|^2$$

Therefore,

$$SNR = \frac{P_S}{P_{N_c}} = \frac{\left|\int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT}df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}\left|C(f)H(f)\right|^2 df}$$

Next we will use the Schwartz inequality, just as we did for the white noise case. In the present case, however, notice that the denominator has a factor *C(f)* while the numerator has no such factor. Thus, we will introduce *C(f)*/*C(f)* into the numerator, which will make it possible to cancel the *C(f)* term in the denominator:

$$SNR = \frac{\left|\int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT}df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}\left|C(f)H(f)\right|^2 df} = \frac{\left|\int_{-\infty}^{\infty} G(f)H(f)\frac{C(f)}{C(f)}e^{j2\pi fT}df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}\left|C(f)H(f)\right|^2 df}$$

$$\le \frac{\int_{-\infty}^{\infty}\left|\frac{G(f)}{C(f)}e^{j2\pi fT}\right|^2 df\int_{-\infty}^{\infty}\left|C(f)H(f)\right|^2 df}{\frac{N_0}{2}\int_{-\infty}^{\infty}\left|C(f)H(f)\right|^2 df} = \frac{2}{N_0}\int_{-\infty}^{\infty}\left|\frac{G(f)}{C(f)}\right|^2 df$$

Equality (achieving maximum SNR) holds when

$$k\left(\frac{G(f)}{C(f)}e^{j2\pi fT}\right)^* = C(f)H(f)$$

$$H(f) = \frac{k\,G^*(f)e^{-j2\pi fT}}{\left|C(f)\right|^2}$$

where *k* is an arbitrary gain factor.