100pts, 110minutes

Student Name: Midterm Solution

Student ID: ____________________

Name of the student sitting to your left: ____________________

Name of the student sitting to your right: ____________________

QUESTION	SCORE	FULL SCORE
1		
2		
3		
4		
TOTAL		100

Note: Please be sure to clearly indicate your answers, and to show how you obtained them.

1. (30 points total)

a. (20 points) As discussed in class, the continuous Fourier transform of a continuous, aperiodic signal $x(t)$ is defined as:

$$X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

Suppose that $x(t)$ is complex and is expressed as $x(t)=x_r(t)+jx_i(t)$ where $x_r(t)$ and $x_i(t)$ are both real functions. Suppose further that $x_r(t)$ is

$$x_r(t)=\text{rect}(t)=\begin{cases}1 & -\frac{1}{2}\le t\le\frac{1}{2}\\ 0 & \text{otherwise}\end{cases}$$

In addition, it is known that $X(f)=0$ for $f<0$.

Find the real and imaginary parts of $X(f)$ for $f>0$. For your answer give separate expressions (valid for $f>0$) for the real and imaginary parts of $X(f)$, $\text{Re}\{X(f)\}$ and $\text{Im}\{X(f)\}$ respectively.

Solution:

There are several ways to solve this problem. One way is to write the Fourier transform expression and exploit the information provided in the problem statement.

$$X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt=\int_{-\infty}^{\infty}\left(x_r(t)+jx_i(t)\right)e^{-j2\pi ft}dt$$

$$=\int_{-\infty}^{\infty}\left(x_r(t)+jx_i(t)\right)\left(\cos 2\pi ft-j\sin 2\pi ft\right)dt$$

$$\text{Re}\{X(f)\}=\int_{-\infty}^{\infty}\left(x_r(t)\cos 2\pi ft+x_i(t)\sin 2\pi ft\right)dt\ .$$

$$\text{Im}\{X(f)\}=-\int_{-\infty}^{\infty}\left(x_r(t)\sin 2\pi ft-x_i(t)\cos 2\pi ft\right)dt$$

Imposing the condition $X(f)=0$ for $f<0$, gives the following relationships.
For $f<0$,

$$\int_{-\infty}^{\infty}\left(x_r(t)\cos 2\pi ft+x_i(t)\sin 2\pi ft\right)dt=0$$

and $\int_{-\infty}^{\infty}\left(x_r(t)\sin 2\pi ft-x_i(t)\cos 2\pi ft\right)dt=0.$

$\Updownarrow$

Thus, for $f<0$,

$\int_{-\infty}^{\infty} x_r(t)\cos 2\pi ftdt = -\int_{-\infty}^{\infty} x_i(t)\sin 2\pi ftdt$ and

$\int_{-\infty}^{\infty} x_r(t)\sin 2\pi ftdt = \int_{-\infty}^{\infty} x_i(t)\cos 2\pi ftdt$

$\Updownarrow$

For $f > 0$, we can use the fact that $\cos(-x) = \cos x$ and $\sin(-x) = \sin x$. This gives

$\int_{-\infty}^{\infty} x_r(t)\cos 2\pi ftdt = \int_{-\infty}^{\infty} x_i(t)\sin 2\pi ftdt$ and

$\int_{-\infty}^{\infty} x_r(t)\sin 2\pi ftdt = -\int_{-\infty}^{\infty} x_i(t)\cos 2\pi ftdt$

Therefore, for $f > 0$,

$$\mathrm{Re}\{X(f)\} = 2\int_{-\infty}^{\infty} x_r(t)\cos 2\pi ftdt = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2\pi ftdt = 2\frac{1}{2\pi f}\sin 2\pi ft\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{2\sin \pi f}{\pi f}$$

$\mathrm{Im}\{X(f)\} = -2\int_{-\infty}^{\infty} x_r(t)\sin 2\pi ftdt = 0$ since $x_r(t)$ is even and $\sin 2\pi ft$ is odd.

Solution 2:

A second way of solving this directly exploits the fact that a real function has a transform which has a real even part and an odd imaginary part. This type of symmetry is called "Hermitian". Similarly, a purely imaginary function has a transform which has an imaginary even part and a real odd part. This type of symmetry can be called "Antihermitian". A general complex function is the sum of a real part and an imaginary part, so a general transform of a complex function is the sum of a Hermitian part and an antiHermitian part. In other words, given $x(t) = x_r(t) + jx_i(t)$, then

$X(f) = X_H(f) + X_A(f)$ where $X_H(f)$ is the Fourier transform of $x_r(t)$, and $X_A(f)$ is the Fourier transform of $jx_i(t)$. In addition, by definition $X_H(f) = X_H^*(-f)$ and $X_A(f) = -X_A^*(-f)$.

As noted in the problem statement, $X(f) = X_H(f) + X_A(f) = 0$ for $f < 0$. So, $X_H(f) = -X_A(f)$ in this range.

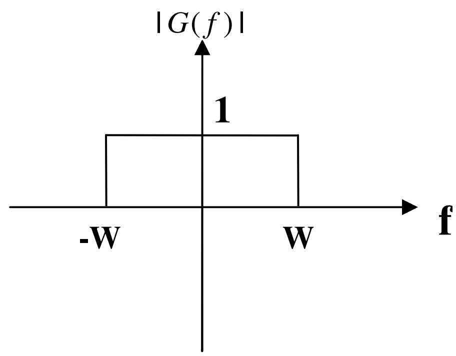
Applying this in combination with the definition of the symmetry relationships gives $X_A(f_1) = -X_A^*(-f_1) = X_H^*(-f_1) = X_H(f_1)$ for any $f_1 > 0$.

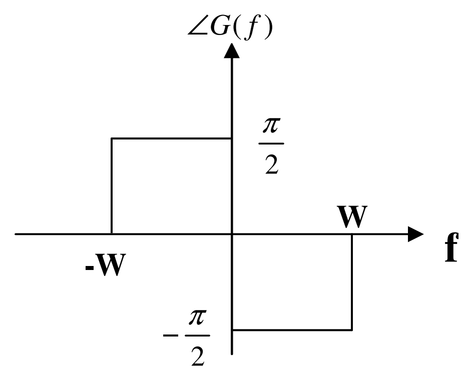
The first equality above is from the definition of Antihermitian symmetry. The second equality is from $X(f) = X_H(f) + X_A(f) = 0$ for $f < 0$ since $-f_1 < 0$. Third equality is from the definition of Hermitian symmetry. These equalities are true for any positive frequency *f*.

Therefore, for $f > 0$, $X(f) = X_H(f) + X_A(f) = 2X_H(f) = 2\frac{\sin \pi f}{\pi f}$, which is purely real.
Thus, for $f > 0$

$\text{Re}\{X(f)\} = \frac{2\sin \pi f}{\pi f}$ and $\text{Im}\{X(f)\} = 0$.

b. (10 points) Suppose $G(f)$ is the Fourier transform of $g(t)$. $G(f)$ is complex, with the magnitude shown in the left figure below and the phase shown in the right figure. Find $g(t)$, the inverse Fourier transform $G(f)$.





Solution:

We can write $G(f)$ as the following:

$$G(f) = j\left(\text{rect}\left(\frac{f + W/2}{W}\right) - \text{rect}\left(\frac{f - W/2}{W}\right)\right).$$

Now using shifting property of Fourier transform and the fact that inverse Fourier transform of a rectangular function is a *sinc* function, would lead us to:

$$g(t) = j\left(W\text{sinc}(Wt)e^{-j2\pi\frac{W}{2}t} - W\text{sinc}(Wt)e^{j2\pi\frac{W}{2}t}\right) = jW\text{sinc}(Wt)\left(e^{-j2\pi\frac{W}{2}t} - e^{j2\pi\frac{W}{2}t}\right)$$

$$\therefore g(t) = 2W\text{sinc}(Wt)\sin(\pi Wt)$$

2. (25 points total)

An AM system transmits a waveform of the form:

$$s(t) = A_c(1 + k_a m(t))\cos(2\pi f_c t)$$

a. (5 points) Assume $|m(t)| \leq 2$ and $k_a > 0$. What is the range of values for k_a that will ensure that over-modulation does not occur?

<u>Solution:</u>

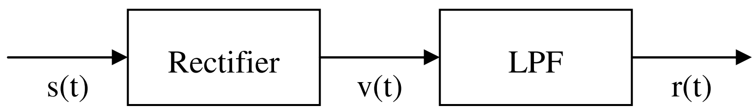
We need $1 + k_a m(t) \geq 0$, so $1 - 2k_a \geq 0$, and then $k_a \leq 1/2$

2. (continued)

b. (20 points) Consider a receiver that uses a full-wave rectifier followed by an ideal lowpass filter, as shown below. The output of the rectifier is the absolute value of the input. The LPF has a gain of 1.0 in the passband and a cutoff frequency of $2f_m$. Let $A_c = 1$, $k_a = 1/4$ and $m(t) = 2\cos(2\pi f_m t)$, and assume that $f_c >> f_m$.

Provide clearly labeled plots of $S(f)$ and $V(f)$, the Fourier transforms of the signals $s(t)$ and $v(t)$ respectively.

State the maximum and minimum values of r(t), the time domain signal at the output of the LPF.

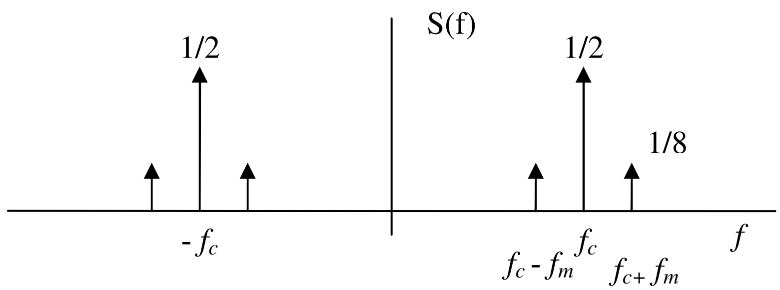


Solution:

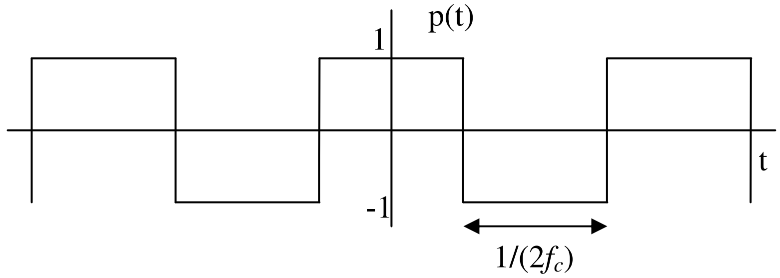
For $s(t)$, the transform is straightforward since this is DSB-TC of a cosinusoidal message signal:

$$S(f) = \frac{A_c}{2}[\delta(f - f_c) + \delta(f + f_c) + k_a M(f - f_c) + k_a M(f + f_c)]$$
$$= \frac{A_c}{2}[\delta(f - f_c) + \delta(f + f_c) + \frac{1}{4}\delta(f - f_c - f_m) + \frac{1}{4}\delta(f - f_c + f_m)$$
$$+ \frac{1}{4}\delta(f + f_c - f_m) + \frac{1}{4}\delta(f + f_c + f_m)]$$

Since $A_c = 1$, this gives the following:



Because $s(t)$ has alternating negative and positive intervals, its absolute value $v(t)$ can be expressed as the product of $s(t)$ and a square wave $p(t)$ with period $1/f_c$ and amplitudes +1 and -1, as shown next.



We can write $p(t)$ as

$$p(t) = [2\text{rect}(2f_c t) * \sum_n \delta(t - n/f_c)] - 1$$

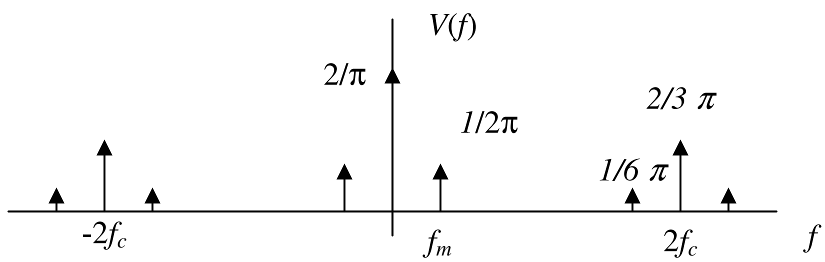
so that

$$P(f) = \frac{1}{f_c}\text{sinc}\left(\frac{f}{2f_c}\right) f_c \sum_n \delta(f - nf_c) - \delta(f) = \sum_{n \neq 0} \text{sinc}(n/2)\delta(f - nf_c)$$

Since $v(t) = p(t)s(t)$, we have that $V(f) = P(f) * S(f)$, so finally,

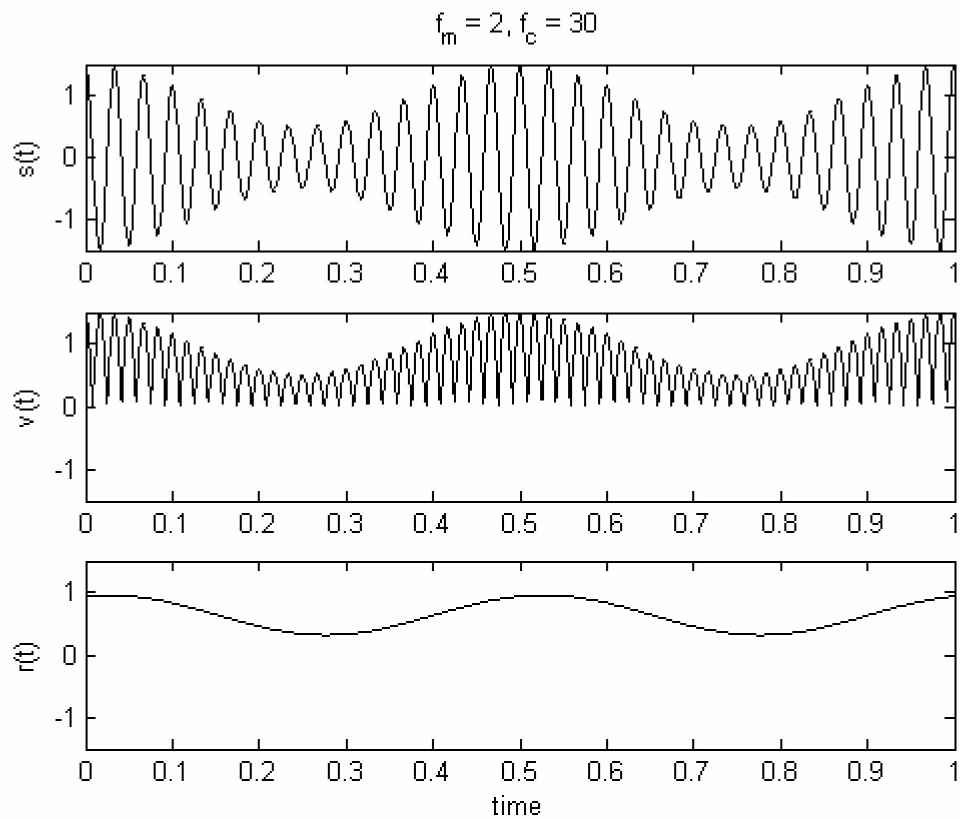
$$V(f) = \frac{A_c}{2}\sum_{n\neq 0} \text{sinc}(n/2)[\delta(f-(n+1)f_c) + \delta(f-(n-1)f_c) + \frac{1}{4}\delta(f-(n+1)f_c - f_m)$$
$$+ \frac{1}{4}\delta(f-(n+1)f_c + f_m) + \frac{1}{4}\delta(f-(n-1)f_c - f_m) + \frac{1}{4}\delta(f-(n-1)f_c + f_m)]$$

Since A_c =1, the plot of $V(f)$ is as follows:



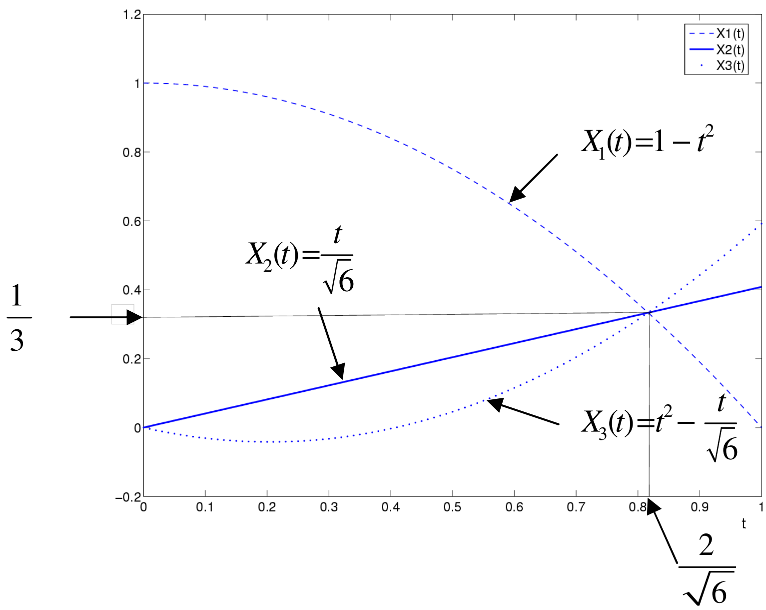
The minimum and maximum values of r(t) can then be obtained by noting that the LPF will pass only the three delta functions near the origin, leaving a time domain signal $r(t) = \frac{2}{\pi} + \frac{1}{\pi}\cos 2\pi f_m t$, which will have maximum $\frac{2}{\pi} + \frac{1}{\pi}$ and minimum $\frac{2}{\pi} - \frac{1}{\pi}$.

The signals s(t), v(t), and r(t) are shown below:



3. (15 points total)

Consider a random process that has three possible realizations as shown below. Realization $X_1(t)$ occurs with probability P_1, realization $X_2(t)$ occurs with probability P_2 and realization $X_3(t)$ occurs with probability $1-P_1-P_2$.



a. (8 points) Find (and simplify to the extent possible) an expression for $E[X(t)]$ valid for the range $0 \le t \le 1$.

<u>Solution:</u>

Let $P_3 = 1 - P_1 - P_2$.

$$E[X(t)] = P_1(1-t^2) + P_2(\frac{t}{\sqrt{6}}) + P_3(t^2 - \frac{t}{\sqrt{6}})$$

$$= t^2(-P_1 + P_3) + \frac{t}{\sqrt{6}}(P_2 - P_3) + P_1$$

$$= t^2(1 - 2P_1 - P_2) + \frac{t}{\sqrt{6}}(-1 + P_1 + 2P_2) + P_1$$

3. (continued)

b. (7 points) Find the autocorrelation $R(t_1, t_2)$ for $t_1 = 0,\ t_2 = \frac{2}{\sqrt{6}}$.

Solution:

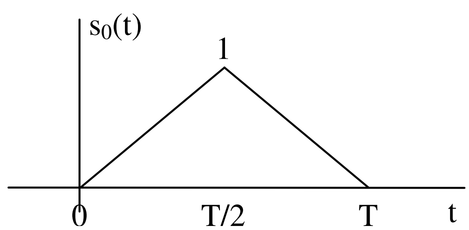
$$R_x(0, \frac{2}{\sqrt{6}}) = E[X(0)X(\frac{2}{\sqrt{6}})] = E[\frac{1}{3}X(0)] = \frac{1}{3}E[X(0)] = \frac{P_1}{3}$$

4. (30 points total)

Consider a baseband binary PAM system that transmits a waveform of the form:

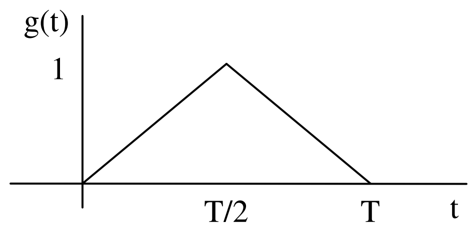
$$s(t) = \sum_{n=-\infty}^{+\infty} a_n s_0(t - nT)$$

where a_n takes values +1 and -1 with equal probability, and $s_0(t)$ has the shape as shown below.



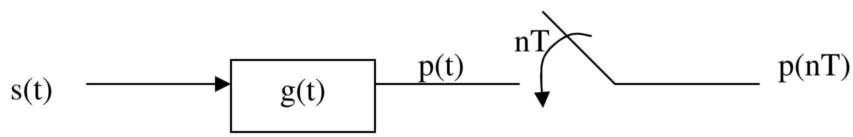
The matched filter for such a system is:

$$g(t) = s_0(T-t) = \begin{cases} 2t/T & \text{if } 0 \le t \le T/2 \\ 2(T-t)/T & \text{if T/2} \le t \le T \\ 0 & \text{otherwise} \end{cases}$$



4. (continued)

Assume now that the signal *s(t)* is supplied to the input of the matched filter, and the output of the matched filter, $p(t)$ is sampled at integer multiples of *T,* as shown in the figure below:



a. (10 points) Compute the possible values of $p(nT)$ produced by sampling the matched filter output at times *nT*. Assume for this part of the problem that there is no noise present.

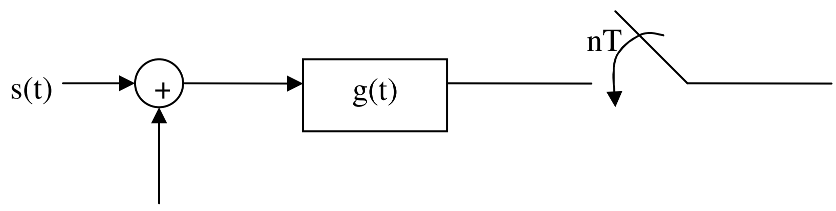
Solution:

In general, $p(t) = g(t) * s(t)$. Since all pulses in *s(t)* have identical shape and are non-overlapping upon arrival at the matched filter input, the matched filter output at the sampling times will be either $\int_0^T s_0^2(t)dt$ or $-\int_0^T s_0^2(t)dt$ depending on whether the transmitted pulse was positive or negative.

Therefore,

$$p(nT) = \pm 2\int_0^{T/2} \left(\frac{2t}{T}\right)^2 dt = \pm 2\frac{1}{3}\left(\frac{4t^3}{T^2}\right)\Bigg|_{t=T/2} = \pm\frac{T}{3}$$

b. (10 points) Now assume that additive white Gaussian noise with power spectral density $\frac{N_0}{2}$ is added prior to the matched filter *g(t)*, as shown below.



Let $\hat{n}(t)$ be the filtered noise output. Compute the output noise power $E[\hat{n}^2(t)]$ measured at the sampling instants *nT*. .

Solution:

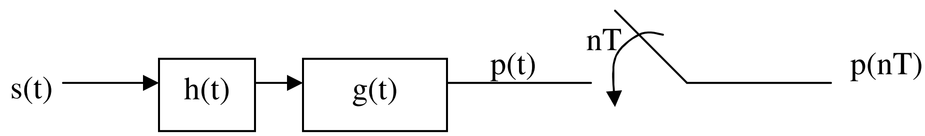
For a matched filter receiver the noise power is

$$E[\hat{n}^2(T)] = \frac{N_0}{2}\int_{-\infty}^{\infty} |G(f)|^2 \, df = \frac{N_0}{2}\int_{-\infty}^{\infty} |g(t)|^2 \, dt$$
$$= \frac{N_0}{2}\int_{-\infty}^{\infty} s_0^{\,2}(T-\tau)d\tau = \frac{N_0}{2}\frac{T}{3} = \frac{N_0 T}{6}$$

.

4. (continued)

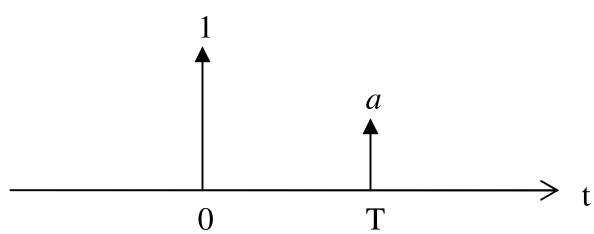
c. (10 points) Now assume that the signal s(t) is transmitted over a channel with frequency response $H(f) = 1 + ae^{-j2\pi fT}$ where a is a constant in the range $0 \le a \le 1$. After transmission over the channel, the matched filter *g(t)* is applied. Assume there is no noise. This system is shown below:



What values can the matched filter outputs $p(nT)$ have? For full credit your answer must give all possible values.

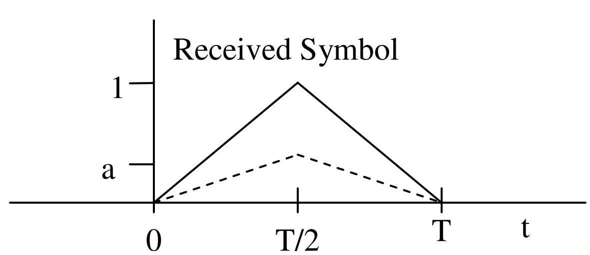
Solution:

This is a channel which has the following impulse response:



Thus, after the channel, each symbol has added to it the previous symbol multiplied by a. In other words, the signal at the output of the channel is $s(t) + as(t-T)$. Depending on the value of the current and previous symbols, there are four combinations. Without any loss of generality, we can consider the outcomes for the pulse starting at t=0. That pulse is shown in solid lines in the figure below. The previous pulse, delayed by T, is shown using dotted lines. The total signal at the channel output (before the matched filter) will be the sum. In each case, the solution from part b can be used to find the matched filter output.

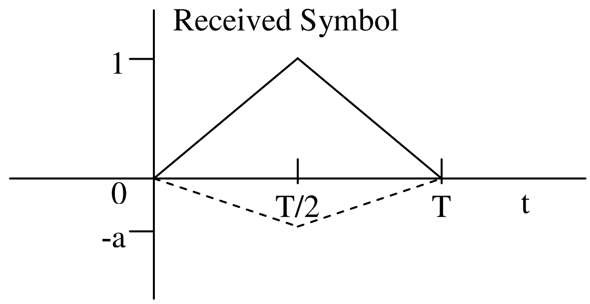
Case 1



$p(t) = g(t) * [s_0(t) + a\, s_0(t)]$

Sampled matched filter output = $(1+a)\frac{T}{3}$

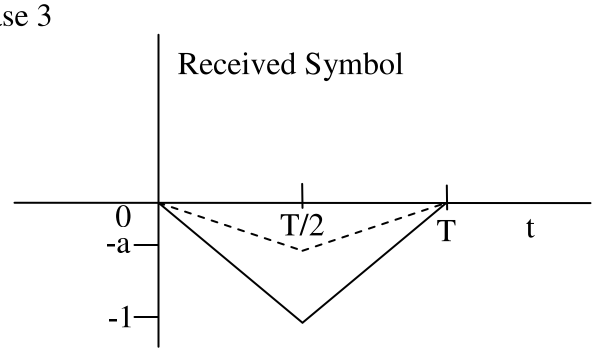
Case 2



$$p(t) = g(t) * [s_0(t) - a\, s_0(t)]$$

Sampled matched filter output = $(1-a)\frac{T}{3}$

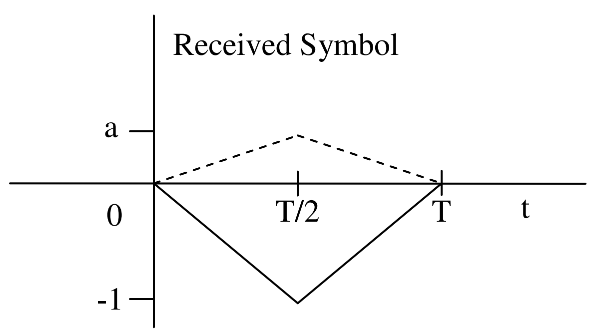
Case 3



$$p(t) = g(t) * [-s_0(t) - a\, s_0(t)]$$

Sampled matched filter output = $(-1-a)\frac{T}{3}$

Case 4



$$p(t) = g(t) * [-s_0(t) + a\, s_0(t)]$$

Sampled matched filter output = $(-1+a)\frac{T}{3}$