100 pts

Student Name: Exam Solutions

Student ID: ____________________

Name of the student sitting to your left: ____________________

Name of the student sitting to your right: ____________________

Question	Score	Full score
1		
2		
3		
4		
Total		

Note: For all problems please circle or otherwise clearly indicate your answers!
The use of calculators or other electronic devices with calculator-like functionality is not permitted on this test.

1.Signal Space (30 points)

Consider a transmission scheme involving the following 8 waveforms, transmitted over an interval of length T (for example $0 \le t \le T$):

$$x_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$$

$$x_2(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t + \frac{\pi}{2}\right)$$

$$x_3(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t + \pi)$$

$$x_4(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t + \frac{3\pi}{2}\right)$$

$$x_5(t) = \sqrt{2}\sqrt{\frac{2}{T}} \cos\left(2\pi f_c t + \frac{\pi}{4}\right)$$

$$x_6(t) = \sqrt{2}\sqrt{\frac{2}{T}} \cos\left(2\pi f_c t + \frac{3\pi}{4}\right)$$

$$x_7(t) = \sqrt{2}\sqrt{\frac{2}{T}} \cos\left(2\pi f_c t + \frac{5\pi}{4}\right)$$

$$x_8(t) = \sqrt{2}\sqrt{\frac{2}{T}} \cos\left(2\pi f_c t + \frac{7\pi}{4}\right)$$

a) (5 points) Specify an appropriate set of normalized basis functions

Answer:

$$x_1(t)=\sqrt{\frac{2}{T}}\cos(2\pi f_c t)$$

$$x_2(t)=\sqrt{\frac{2}{T}}\cos\left(2\pi f_c t+\frac{\pi}{2}\right)=\sqrt{\frac{2}{T}}\left[-\sin(2\pi f_c t)\right]$$

$$x_3(t)=\sqrt{\frac{2}{T}}\cos(2\pi f_c t+\pi)=\sqrt{\frac{2}{T}}\left[-\cos(2\pi f_c t)\right]$$

$$x_4(t)=\sqrt{\frac{2}{T}}\cos\left(2\pi f_c t+\frac{3\pi}{2}\right)=\sqrt{\frac{2}{T}}\left[\sin(2\pi f_c t)\right]$$

$$x_5(t)=\sqrt{2}\sqrt{\frac{2}{T}}\cos\left(2\pi f_c t+\frac{\pi}{4}\right)=\sqrt{\frac{2}{T}}\left[\cos(2\pi f_c t)-\sin(2\pi f_c t)\right]$$

$$x_6(t)=\sqrt{2}\sqrt{\frac{2}{T}}\cos\left(2\pi f_c t+\frac{3\pi}{4}\right)=\sqrt{\frac{2}{T}}\left[-\cos(2\pi f_c t)-\sin(2\pi f_c t)\right]$$

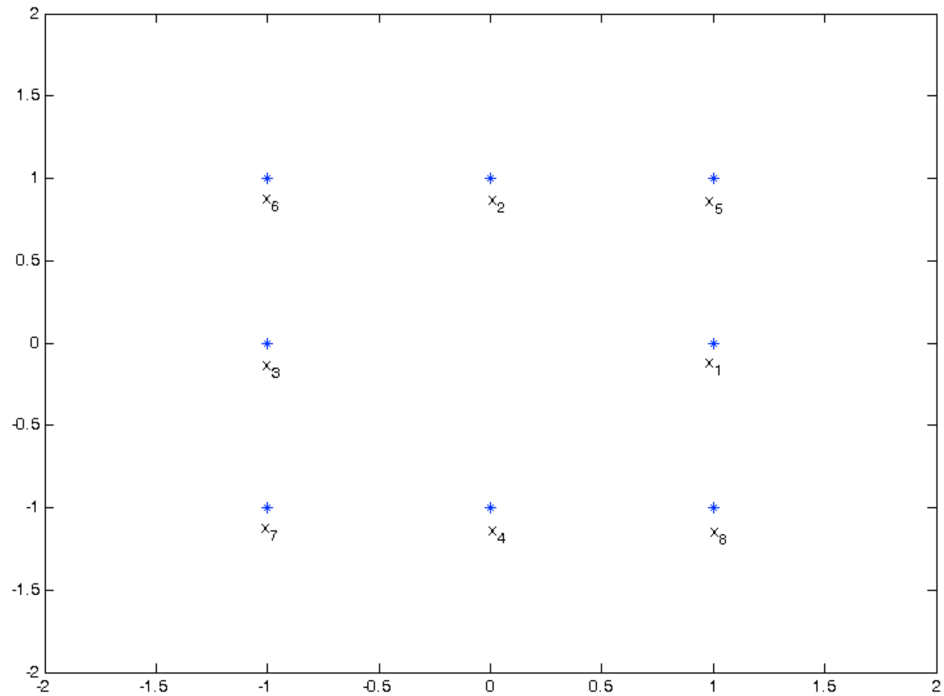
$$x_7(t)=\sqrt{2}\sqrt{\frac{2}{T}}\cos\left(2\pi f_c t+\frac{5\pi}{4}\right)=\sqrt{\frac{2}{T}}\left[-\cos(2\pi f_c t)-\sin(2\pi f_c t)\right]$$

$$x_8(t)=\sqrt{2}\sqrt{\frac{2}{T}}\cos\left(2\pi f_c t+\frac{7\pi}{4}\right)=\sqrt{\frac{2}{T}}\left[\cos(2\pi f_c t)+\sin(2\pi f_c t)\right]$$

Therefore, choose $\phi_1(t)=\sqrt{\frac{2}{T}}\cos(2\pi f_c t)$ and $\phi_2(t)=-\sqrt{\frac{2}{T}}\sin(2\pi f_c t)$ as orthonormal basis functions.

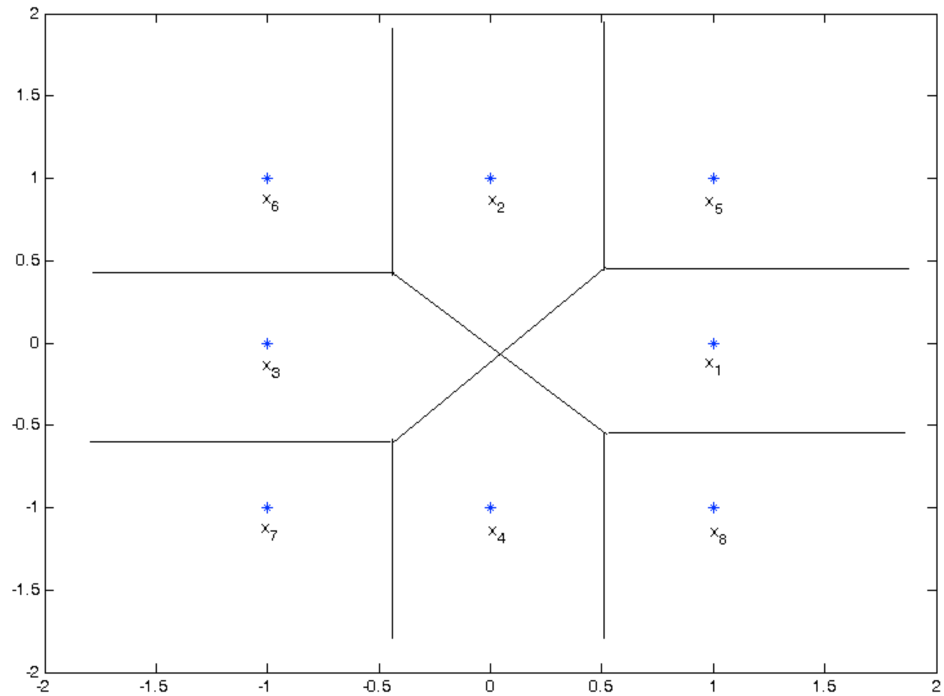
b) (7 points) Plot the constellation in signal space, making sure the location of each symbol is clear.

Answer:



c) (10 points) The transmitted signal is corrupted by AWGN with power spectral density $N_0/2$, and the receiver performs maximum likelihood detection. Assume each symbol is transmitted with equal probability. Provide a signal space plot showing the locations of the symbols as well as the decision boundaries.

Answer:



d) (8 points) Compute the tightest possible union bound on the overall probability of symbol error for this constellation assuming AWGN and that all symbols have equal probability. Your answer should be a single number (or maximally simplified expression in terms of the Q or erfc functions) that provides the average probability of symbol error, given that all symbols are equally likely to be transmitted.

Answer:

With respect to error computations, there are two different types of symbols in this constellation: the "corner" symbols and the "middle" symbols. For the corner symbols, the union bound involves two neighbors, and for the middle symbols, the union bound involves four neighbors (because the associated decision region has four sides). Since there are 4 corner symbols and 4 middle symbols, the union bounds from these two types of symbols should be averaged to get the overall answer.

$$P\left(error \mid x_i, i \le 4\right) \le 2Q\left(\frac{1}{2\sigma}\right) + 2Q\left(\frac{\sqrt{2}}{2\sigma}\right) = 2Q\left(\sqrt{\frac{1}{2N_0}}\right) + 2Q\left(\sqrt{\frac{1}{N_0}}\right)$$

$$P\left(error \mid x_i, i \ge 5\right) \le 2Q\left(\frac{1}{2\sigma}\right) = 2Q\left(\sqrt{\frac{1}{2N_0}}\right)$$

$$P\left(error\right) = 4 \times \frac{1}{8} \times P\left(error \mid x_i, i \le 4\right) + 4 \times \frac{1}{8} \times P\left(error \mid x_i, i \ge 5\right)$$

$$\le \frac{1}{2}\left[2Q\left(\sqrt{\frac{1}{2N_0}}\right) + 2Q\left(\sqrt{\frac{1}{N_0}}\right)\right] + \frac{1}{2}\left[2Q\left(\sqrt{\frac{1}{2N_0}}\right)\right]$$

$$= 2Q\left(\sqrt{\frac{1}{2N_0}}\right) + Q\left(\sqrt{\frac{1}{N_0}}\right)$$

$$= erfc\left(\sqrt{\frac{1}{4N_0}}\right) + \frac{1}{2}\ erfc\left(\sqrt{\frac{1}{2N_0}}\right)$$

2. Receiver Decision (25 points)

Consider a binary transmission system transmitting $X = \pm 1$ with equal probability. In contrast with most of the systems we have studied in class, in this system the noise depends on the signal that is transmitted. More specifically, when $X = +1$ is transmitted, the noise is AWGN with variance $\sigma^2=4$. When $X = -1$ is transmitted, the noise is AWGN with variance $\sigma^2=1$. The formula for a Gaussian distribution with mean μ and standard deviation σ is as the follows:

$$f\left(v\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v-\mu)^2}{2\sigma^2}}$$

At the input to the decision module of the receiver (after demodulation, etc.), the value of the signal is denoted by v. In the absence of noise, v would of course either be +1 or -1. In the presence of noise as described above, however, the received signal is drawn from one of two Gaussian pdfs as described above. Assume that a maximum likelihood receiver is used. The receiver determines, based on the value v, whether the transmitted value was more likely to have been +1 or -1.

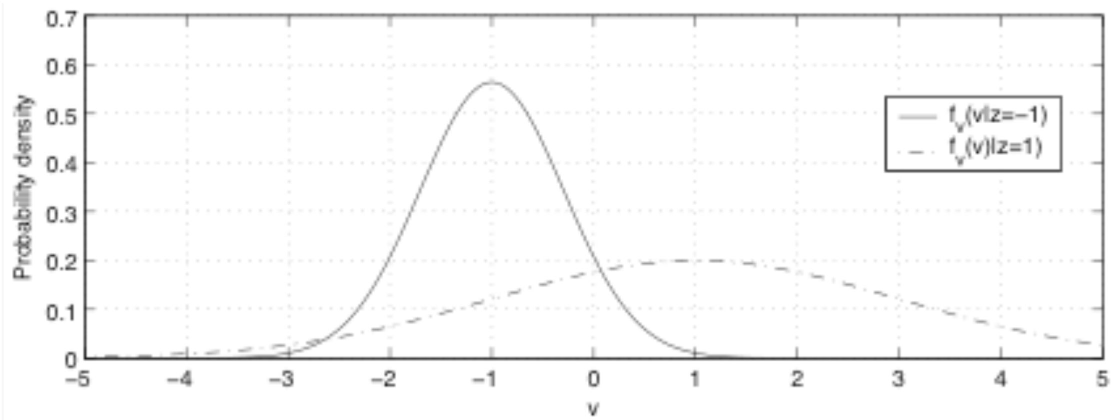
Suppose that in association with a particular transmission, v is found to be -5. Based on knowledge of the fact that v is -5, suppose that a decision is made that +1 was transmitted.

a) (10 points) Is the decision that +1 was transmitted most likely to be right or wrong? Your answer must be supported by a proof.

b) (15 points) What is the probability that this is the wrong decision?

Answer

A plot of the two pdfs associated with the respective transmitted signals is below:



If the receiver decides that the transmitted signal was +1, this decision can either be correct or incorrect. The probabilities of being correct or incorrect depend on the values of the respective pdfs at v= -5.

The probability that a decision that +1 was transmitted is an incorrect decision is:

$$p(wrong) = \frac{p(-1 transmitted)}{p(1 transmitted) + p(-1 transmitted)}$$

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(-5+1)^2}{2}}}{\frac{1}{\sqrt{2\pi 4}} e^{-\frac{(-5-1)^2}{2\times 4}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(-5+1)^2}{2}}}$$

$$= \frac{e^{-8}}{\frac{1}{2} e^{-4.5} + e^{-8}}$$

Solution for part a) As can be seen from the above, it is most likely (by far) that the decision of +1 is correct.
Solution for part b) See computation for incorrect decision probability above.

3. OFDM (20 points)

a) (10 points) Consider an OFDM transmission scheme in which $N = 48$ subcarriers are used for transmission. OFDM symbols are transmitted at a rate of 250K symbols/second. Information bits are first subject to channel coding, and the resulting coded bits output from the channel coder are mapped into a modulation scheme. The possible channel coding rates are: $r = 1/2, 2/3, 3/4$. The possible modulation schemes are: BPSK, QPSK, 16-QAM, 64-QAM. Determine the maximum number of information bits per second that can be transmitted using this scheme.

Answer

The maximum data occurs when the coding rate ¾ and 64-QAM is used on all subcarriers. Each OFDM symbol will have a duration of 4 microseconds.

$$R_{\max} = 48 \text{ subcarriers}.\frac{3 \text{ info bits}}{4 \text{ block bits}}.\frac{6 \text{ coded bits}}{\text{subcarrier symbol}}.\frac{1 \text{ subcarrier symbols}}{4(10)^{-6} \text{ seconds}}$$

$$= 54\text{Mbps}$$

b) (10 points) In an OFDM system, the baseband signal corresponding to subcarrier *i* can be expressed for the case of rectangular pulse shapes as $x_i(t) = e^{j2\pi it(\Delta f)}$ where $\Delta f = \frac{1}{T_N}$ is the frequency separation between subcarriers. Similarly the baseband signal for subcarrier *i*+1 can be written as $x_{i+1}(t) = e^{j2\pi(i+1)t(\Delta f)}$. The lowest frequency subcarrier (*i*=0) is at frequency zero. For subcarrier *i*=1 the frequency is $\frac{1}{T_N}$, and for subcarrier *m* the frequency is $\frac{m}{T_N}$.

In an ideal case, the signal at subcarrier *i* does not experience any interference from the adjacent subcarrier *i*+1 because the respective signals are orthogonal (thus the name OFDM). In other words, the integral below providing the interence signal

$$I = \int_0^{T_N} x_i(t) x_{i+1}^*(t)\,dt$$

will normally be zero. Stated another way, the projection of the term x_{i+1} onto x_i is zero. Suppose, however, that subcarrier *i*+1, which should be at frequency $\frac{i+1}{T_N}$, is instead transmitted at $\frac{i+1+\delta}{T_N}$. In this case, the projection of the term x_{i+1} onto x_i will not be zero, and there will be some signal power appearing at the receiver for subchannel *i* that originated in subchannel *i+1*. Find a maximally simplified expression for this power.

Answer

The projection of the term x_{i+1} onto x_i is given by

$$I = \int_0^{T_N} e^{j2\pi it/T_N} e^{-j2\pi(i+1+\delta)t/T_N}\, dt$$

$$= \int_0^{T_N} e^{-j2\pi(1+\delta)t/T_N}\, dt = \frac{T_N\left(1-e^{-j2\pi(1+\delta)}\right)}{j2\pi(1+\delta)}$$

$$= T_N e^{-j\pi(1+\delta)} \frac{\left(e^{j\pi(1+\delta)} - e^{-j\pi(1+\delta)}\right)}{j2\pi(1+\delta)}$$

$$= T_N e^{-j\pi(1+\delta)} \operatorname{sinc}(1+\delta)$$

The power is expressed as

$$P = \frac{|I|^2}{T_N} = T_N \operatorname{sinc}^2(1+\delta)$$

4. Linear Block Codes (25 points)

a) (5 points) In a linear block code, if *c* is known to be a valid codeword, what is the result of cH^T, where *H* is the parity check matrix? Your answer should specify whether the result of multiplying cH^T is a scalar, vector or a matrix. If it is a vector or matrix specify its dimensions. Also specify the contents of cH^T when *c* is a valid codeword.

Answer:

The parity check equation is $cH^T = 0$. Where 0 is a vector of size *n-k* containing all zeros.

b) (10 points) Consider a systematic (7,3) code with the following parity check matrix. Note that there are four unknown binary elements, labeled A, B, C, and D below. :

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & A & B & 0 & 0 & 1 & 0 \\ 1 & C & D & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Suppose that $c = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$ is known to be a valid codeword. Provide all the possible valid solutions for:

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

Answer:
This can be solved by enumerating all the combinations of missing values in column 2 and column 3 that satisfy $cH^T = 0$. By analyzing the nonzero elements of the received codeword, it is only the modulo-2 sum of rows two, three, six and seven of H^T that contribute to producing the all-zero syndrome ensuring c is a valid codeword.
Since there are four missing elements, there are sixteen possible ways to fill these elements with ones and zeros. However, only four of these sixteen cases will produce $cH^T = 0$ given the valid codeword in the problem statement. These four are:

$$\begin{matrix} A & B \\ C & D \end{matrix} = \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}, \quad \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix}, \quad \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}, \text{ or } \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}.$$

c) (10 points) Provide all possible valid solutions of

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

that satisfy both of the following two conditions: 1) the code includes $c = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$ as one of the codewords in the code, and 2) there is no solution for A, B, C, D giving a higher d_{min}. Stated another way, you should find A, B, C, D that result in a code containing $c = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$ and having the maximum possible d_{min}. Recall that d_{min} is the Hamming distance from any codeword to the nearest other codeword in the code.

Answer

c) Given $H = \begin{bmatrix} P^T & I_{n-k} \end{bmatrix}$, we can construct

$$G = \begin{bmatrix} I_k & P \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & A & C \\ 0 & 0 & 1 & 1 & 1 & B & D \end{bmatrix}$$

Since the distances separating codewords do not depend on which codeword is used as the reference, for convenience it is easiest to use the all-zero codeword has a reference. For a given code, d_{min} will be the number of ones in the codeword having the fewest number of ones. Of the four possible valid

$$\begin{matrix} A & B \\ C & D \end{matrix}$$

values, there are two sets of values that ensure that all rows of G have a Hamming weight of at least 4 : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Note that *A*=*D* and *C*=*B* in each of these cases, so $\begin{matrix} A & B \\ C & D \end{matrix}$ is equal to its transpose $\begin{matrix} A & C \\ B & D \end{matrix}$. To demonstrate that these are the only two unique combinations we construct *G* and test whether all linear combinations of the rows of *G* produce a weight-four code.

Therefore, $H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ or alternatively

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$