UCLA Electrical Engineering Department

ECE132A: Communication Systems

MIDTERM EXAMINATION

SET A

Spring 2018

Monday, May 07, 2018

Exam Duration: 1 hr. 30 min.
Total: 50 points

NAME: SOLUTION SET

UID: XXX-XXX-XXX

	1	2	3	4	Total
Marks Obtained	12	12	13	13	50
Maximum Marks	12	12	13	13	50

I understand that academic integrity is highly valued at UCLA. Further, I understand that academic dishonesty such as cheating and plagiarism, are violations of University Policy and will be pursued by the appropriate campus administrator. Finally, my signature below signifies that the work included is my own, and I completed this assignment honestly.

Signature: ______________________________

Instructions:

(i) Books, cheat sheet, formulae sheet, Calculators, Cell Phones, Computer, Laptops, Tablets, Programmable watches, and IPods are **NOT** allowed
(ii) Provide your solutions only within the space provided within this booklet. Your answers must be legible and easy to follow.
(iii) PLEASE MAKE SURE THAT YOU HAVE 12 PAGES OF THIS EXAMINATION BOOKLET.

FORMULAE SHEET

1. Let $Z = X + Y$, where X and Y are independent random variables with distributions (p.d.f) $f_X(x)$ and $f_Y(y)$, respectively. Then $f_Z(z) = f_X(x) \otimes f_Y(y)$, the convolution of the PDFs of X and Y.
2. The distance between two constellation points $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$ is $d = \|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$
3. $Q(a) \triangleq \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$, the tail probability of a standard normal distribution. $Q(-a) = 1 - Q(a)$
4. $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$; $\cos\left(\frac{\pi}{8}\right) = 0.92$; $\sin\left(\frac{\pi}{8}\right) = 0.38$
5. For a Gaussian random variable, $F_X(x) = P(X \le x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = 1 - Q\left(\frac{x-\mu}{\sigma}\right)$, where μ is the mean of X and σ^2 is the variance of X. If X is Gaussian, then $Y = AX$ is also Gaussian.

   Joint Gaussian: $f_X(x_1, x_2, \dots, x_n) = \frac{1}{(2\pi)^{\frac{n}{2}}|\Lambda_X|^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu_X)^T \Lambda_X^{-1}(x-\mu)}$

   Wide sense stationary random process: $E[X(t)] = \mu \ \forall t$ and $R_X(t_1, t_2) = R_X(|t_2 - t_1|) \ \forall t_1, t_2$
6. Probability of error:

   $P(error) = \sum_{i=1}^{M} P_H(H = i) P(e_i)$, where $P(e_i) = P(\hat{H} \ne i | H = i)$
7. Nearest Neighbor bound:

$$P_e(i) = \sum_{\substack{j \ne i \\ j \in N_i}} Q\left(\frac{\|x_i - x_j\|}{2\sigma}\right) \le |N_i| Q\left(\frac{d_{\min}}{2\sigma}\right)$$

   where, $|N_i|$ is the number of nearest neighbors to $\boldsymbol{x}_i$.
8. MAP criteria: $d_{MAP}(y) = \arg\max_j p_{X|Y}(X = x_j | Y = y)$
9. ML criteria: $d_{ML}(y) = \arg\max_j f_{Y|X}(y|x_j) = \arg\min_j (y - x_j)^2$
10. Vector hypothesis:

$$d_{MAP}(y) = \arg\max_j \left\{ \ln p_X(j) - -\frac{1}{2}\frac{\|y - x_j\|^2}{\sigma^2} \right\}$$

$$d_{ML}(y) = \arg\max_j \left\{ -\frac{1}{2}\frac{\|y - x_j\|^2}{\sigma^2} \right\} = \arg\min_j \left\{ \|y - x_j\|^2 \right\}$$

11. Threshold 1:

$$H_0: p_H(H = 0) f_{Y|X}(y|x_0) \ge p_H(H = 1) f_{Y|X}(y|x_1)$$
$$H_1: p_H(H = 0) f_{Y|X}(y|x_0) < p_H(H = 1) f_{Y|X}(y|x_1)$$

   Threshold 2:

$$H_0: p_{X|Y}(X = x_0|y) \ge p_{X|Y}(X = x_1|y)$$
$$H_1: p_{X|Y}(X = x_0|y) < p_{X|Y}(X = x_1|y)$$

12. Bayes' rule:

$$p_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) p_X(x)}{f_Y(y)} = \frac{f_{Y|X}(y|x) p_X(x)}{\sum_j f_{Y|X}(y|x_j) p_X(x_j)}$$

1. *(12 points) Short questions and basic probabilities.*

A. *True or False*. State if the below statements are true or false. Provide justification for the full credit (only 0.5 point for True or False).

a) (1.5 points) Let A and B be two i.i.d. Gaussian random variables with mean μ and variance σ^2. Then, $P[2A + B > 3] = Q\left(\frac{\sqrt{3}(1-\mu)}{\sigma}\right)$.

False. let $Y = 2A + B$ $\quad \mu_Y = 2\mu + \mu = 3\mu$,

$\sigma_Y^2 = 5\sigma^2 \quad P[2A+B>3] = P[Y>3] = Q\left(\frac{3-3\mu}{\sqrt{5}\sigma}\right)$

b) (1.5 points) If X is uniformly distributed random variable on the real-line with a unit variance, then the pdf of X is completely defined.

False. Since we don't know the mean of X, the rectangular pdf can be stuffed arbitrarily and we don't know the pdf completely.

c) (1.5 points) Denote the Gaussian random vector $z = [X \quad Y]^T$. Let X be a Gaussian scalar random variable with $X \sim N(0,1)$ and Y be a Gaussian scalar random variable with $Y \sim N(1,4)$. Then the pdf of Z is defined.

False. For arbitrary Z, we need to know the correlation between X and Y.

d) (1.5 points) Consider a binary hypothesis, where message $H \in \{0,1\}$ is mapped to $X \in \{X_0, X_1\}$, i.e. $X = X_i$ when $H = i$, where $i \in \{0,1\}$. Under any binary hypothesis, observed X such that $p_{X_1}(x) > p_{X_0}(x)$ results in a decision for $H = 1$.

False. $P_X(x)$ is a priori information. For declaring any decision region, we need to evaluate a-posteriori probability such that: $P_{X_0}(x_0) \cdot p(y|x_0) \underset{H_1}{\overset{H_0}{\gtrless}} P_{X_1}(x_1)\, p(y|x_1)$

e) (1 points) The autocorrelation function $R(t)$ of a wide sense stationary process is always non-negative.

False. If there is negative correlation between $X(t_1)$ and $X(t_2)$, then it could be negative.

f) (1 points) A receiver that implements the ML decision rule is always optimal in the sense of minimum symbol error probability.

False. for case where a-priori is not uniform, we use MAP.

g) (1 point) The MAP decision rule is a special case of the ML rule.

False. ML is a special case of MAP.

B. (3 points) A drawer contains red and black socks, when two socks are drawn at random, the probability that both are red is $\frac{1}{2}$. How small is the number of socks in the drawer?

let # of red socks $= r$ & # of black socks $= b$

$\therefore P(\text{both socks are red}) = \frac{r}{r+b} \times \frac{r-1}{r+b-1} = \frac{1}{2} \Rightarrow 2r(r-1) = (r+b)(r+b-1)$

Method 1: Intuitive approach: Let $b=1 \Rightarrow r=3$ and $r+b=4$. [$\because$ for $b=1$, $r=3$]

$2r(r-1) = r(r+1)$ $\Rightarrow r=3$. $\therefore P = \frac{3}{4} \times \frac{2}{3} = 1/2$ & Hence, min # of socks in drawer $= 4$

Method 2: Methodical: Now, $\frac{r}{r+b} > \frac{r-1}{r+b-1}$ $\therefore \left(\frac{r}{r+b}\right)^2 > \frac{1}{2} > \left(\frac{r-1}{r+b-1}\right)^2$

$\Rightarrow \frac{r}{r+b} > \frac{1}{\sqrt{2}} > \frac{r-1}{r+b-1}$ or $(\sqrt{2}+1)b+1 > r > (\sqrt{2}+1)b$, when $b=1$, r in between 2.414 & 3.414. so, $r=3$

2. *(12 points) Random Process, Wide Sense Stationary Process and Irrelevance.*

A. (5 points) Let $\boldsymbol{X} = [X_1 \quad X_2]^T$ be real-valued Gaussian random vector with the following mean vector and covariance matrix:

$$\boldsymbol{\mu_X} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{\Lambda_X} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Now let $\boldsymbol{Y} = \boldsymbol{AX} + B$, where $A = [1 \quad 1]$, $B = 2$. Find $E[Y], Var[Y]$ and $p_Y(y)$.

Solution:

$Y = AX + B$

(a) $E[Y] = A\,E[X] + B = [1 \quad 1]\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 = 3 \ (\mu_Y)$

(b) $Var[Y] = A \Lambda_X A^T = [1 \quad 1]\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7 \ (\sigma_Y^2)$

(c) $p_Y(y) = \frac{1}{\sqrt{2\pi}\sqrt{7}} e^{-\frac{(y-3)^2}{14}}$

B. (4 points) Let A be a non-negative random variable that is independent of any collection of samples $X(t_1), \ldots, X(t_k)$ of a wide sense stationary random process $X(t)$. Is $Y(t) = A + X(t)$ a wide sense stationary process?

Solution (1) $E[Y(t)] = A + \mu_x$: independent of t $\{\because X(t)$ is WSS, so $E[X(t)] = \mu_x\}$

(2) $R_Y(t_1,t_2) = E[(A+X(t_1))(A+X(t_2))] = E[A^2 + AX(t_2) + AX(t_1) + X(t_1)X(t_2)]$

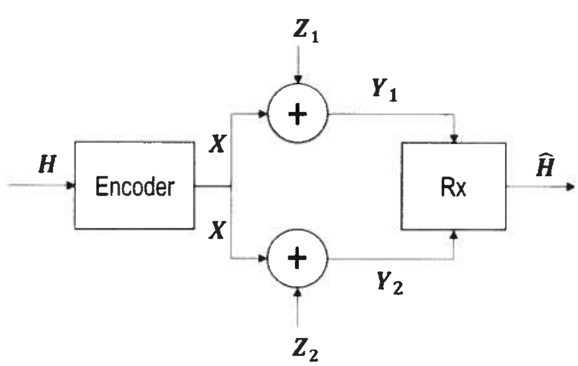
$= A^2 + A\,E[X(t_2)] + A\,E[X(t_1)] + R_X(\tau)$

$\therefore R_Y(t_1,t_2) = A^2 + 2A\mu_x + R_X(\tau)$: only a function of τ

From (1) and (2), we show Y is WSS.

Note: Here we assumed that A is constant. If not, then we can use $E[A]$ and $E[A^2]$. It still shows Y is WSS.

C. (3 points) Consider a communication system below in Figure 1. The data is transmitted through two parallel AWGN channels with the noise Z_1 and Z_2 respectively. The mapped waveform $\boldsymbol{X}$ is independent of the noise $\boldsymbol{Z} = [Z_1 \quad Z_2]^T$ i.e, $\boldsymbol{X}$ is independent of both Z_1 and Z_2. The noise vector $\boldsymbol{Z}$ is jointly Gaussian with the distribution $[Z_1 \quad Z_2]^T \sim N(\boldsymbol{\mu}, \boldsymbol{\Lambda_Z})$, where $\boldsymbol{\Lambda_Z}$ is the full rank covariance matrix and hence, Z_1 and Z_2 are correlated. For such a communication system, show if Y_2 is either irrelevant or not?



$Y_1 = X + Z_1$

$Y_2 = X + Z_2$

Figure 1: Communication System for Problem #2C

For, Y_2 to be irrelevant, we need to show, $H - X - Y_1 - Y_2$ forms Markov chain

$Z_1 \not\perp Z_2$

$$P_{Y_2|Y_1,H}(y_2|y_1,H) = P_{Z_2|Z_1}(y_2 - x|y_1 - x)$$
$$\neq P_{Y_2|Y_1}(y_2|y_1)$$

$\therefore$ Markov chain is not satisfied

Hence, Y_2 is not irrelevant.

3. *(13 points) Signal Constellation.*
   A. (6 points) For minimum Euclidean distance decision rule, i.e.,

$$\arg \min_{m=1,2,3,4} \|x - s_m\|$$

draw the best decision partition for the below signal constellation, where $s_1 = (\sqrt{2}a, 0)$, $s_2 = (0, \sqrt{2}a)$, $s_3 = (-\sqrt{2}a, 0)$ and $s_4 = (0, -\sqrt{2}a)$.

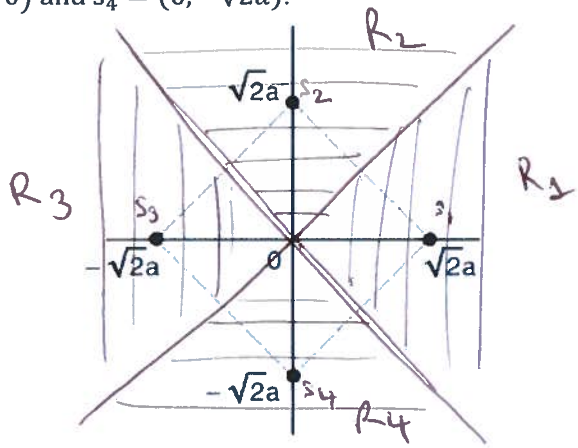


Figure 2: Signal Constellation for Problem #3A.

a) (2 points) Compute the energy of the four signal points.

$$E_{avg} = \frac{4 \cdot (\sqrt{2}a)^2}{4} = 2a^2$$

b) (4 points) Show that for the above signal constellation, the minimum distance decision rule can be reduced to an inner-product decision rule, i.e.,

$$\arg \min_{m=1,2,3,4} \|\boldsymbol{x} - \boldsymbol{s}_m\| = \arg \max_{m=1,2,3,4} \langle \boldsymbol{x}, \boldsymbol{s}_m \rangle$$

$$\|\underline{x} - \underline{s}_m\|^2 = (\underline{x} - \underline{s}_m)^T(\underline{x} - \underline{s}_m) = \underline{x}^T\underline{x} - \underline{x}^T\underline{s}_m - \underline{s}_m^T\underline{x} + \underline{s}_m^T\underline{s}_m$$

$$= \underline{x}^T\underline{x} - 2\underline{x}^T\underline{s}_m + \|\underline{s}_m\|^2$$

For the given signal constellation, $\|\underline{s}_m\|^2$ is constant and hence, independent of m.
Similarly, $\underline{x}^T\underline{x}$ is independent of m.

$$\therefore \arg \min_m \|\underline{x} - \underline{s}_m\| = \arg \min_m \|\underline{x} - \underline{s}_m\|^2$$

$$= \arg \min_m \left[-2\underline{x}^T\underline{s}_m\right] = \arg \max_m \left[\underline{x}^T\underline{s}_m\right]$$

$$= \arg \max_m \langle \underline{x}, \underline{s}_m \rangle$$

B. (7 points) Consider the following signal sets:

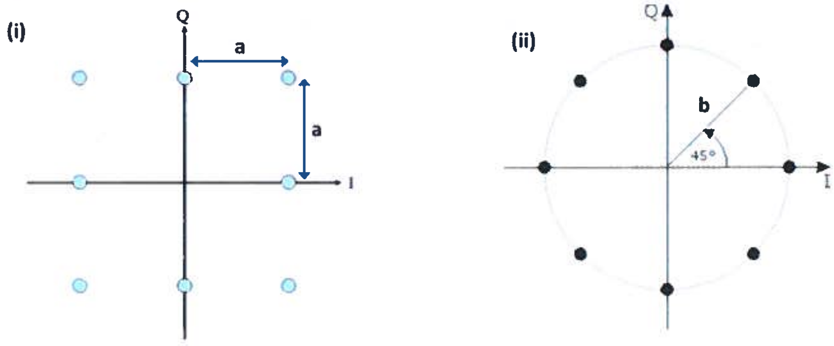
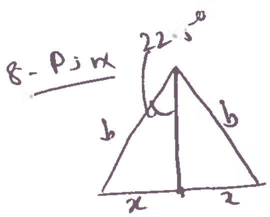


Figure 3: Signal Constellation for Problem #3B.

Use the trigonometric values from Formulae Sheet as needed.

a) (2 points) Express the minimum distance in terms of the average energies.

$$E_{avg,8\text{-}QAM} = \frac{4a^2 + 4(\sqrt{2}a)^2}{8} = \frac{3}{2}a^2 \Rightarrow a = \sqrt{\frac{2}{3}E_{avg}} = d_{min}$$



$$E_{a,8\text{-}PSK} = \frac{8b^2}{8} = b^2 \Rightarrow b = \sqrt{E_{avg}}.$$

$$d_{min} = 2x = 2b\sin(22.5^\circ) = 2\sqrt{E}\sin\frac{\pi}{8} \approx 0.76\sqrt{E_a}$$

b) (5 points) Compare the signal sets in terms of $P(e)$ using nearest neighbor union bound. What do you conclude?

$$P_{e,8\text{-}QAM} \leq 2Q\left(\frac{1}{2\sigma}\sqrt{\frac{2}{3}E_a}\right) = 2Q\left(\frac{1}{2\sigma}0.816\sqrt{E_a}\right)$$

$$P_{e,8\text{-}PSK} \leq 2Q\left(\frac{0.76}{2\sigma}\sqrt{E_a}\right)$$

$$\therefore P_{e,8QAM} < P_{e,8PSK} \quad \left\{ \because x\uparrow,\ Q(x)\downarrow \right.$$

4. *(13 points) Decision Rule and Gram Schmidt Orthogonalization.*

A. (6 points) Consider the waveforms shown in Figure 4 below.

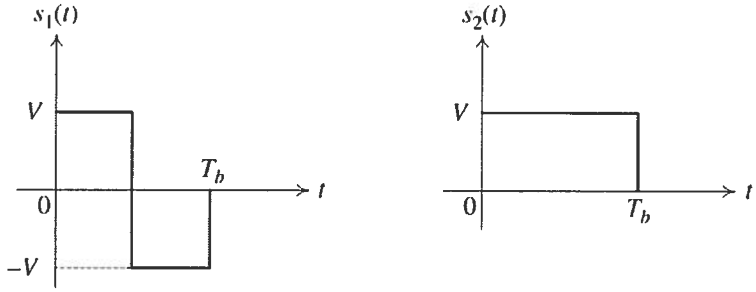


Figure 4: Signals for Problem #4A.

a) (4 points) Find the basis functions using Gram Schmidt Orthogonalization method. Plot the obtained basis functions with respect to time.

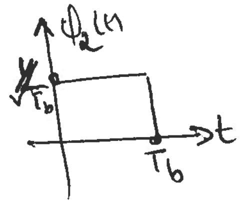
$0 \le t \le T_b$

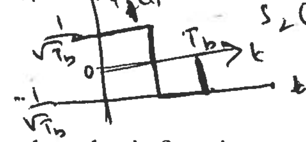
$$\phi_1(t) = \frac{s_1(t)}{\|s_1(t)\|} \; ; \; \|s_1(t)\|^2 = \int_0^{T_b} V^2\,dt = V^2 T_b \Rightarrow \|s_1(t)\| = V\sqrt{T_b} = \|s_2(t)\|$$

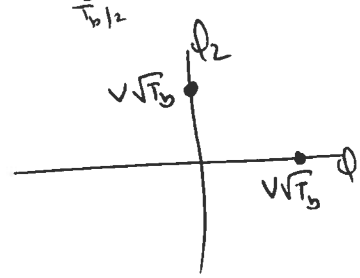
$$g_2(t) = s_2(t) - c_{12}\phi_1(t) \Rightarrow c_{12} = \int_0^{T_b} s_1(t)\,s_2(t)\,dt = \int_0^{T_b/2} V^2\,dt - \int_{T_b/2}^{T} V^2\,dt = 0$$

$$\therefore \phi_2(t) = \frac{s_2(t)}{\|s_2(t)\|}$$

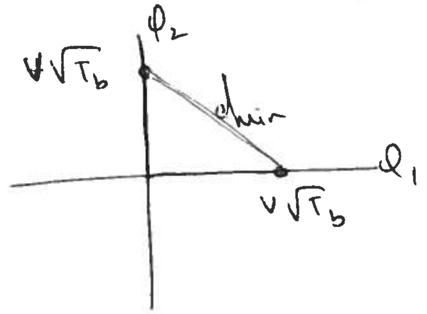
$$\therefore s_1(t) = V\sqrt{T_b}\,\phi_1(t)$$
$$s_2(t) = V\sqrt{T_b}\,\phi_2(t)$$







b) (2 points) Use these basis functions to represent the waveform constellation and determine the minimum distance between the signals.



$$d_{min}^2 = 2V^2 T_b$$

$$\Rightarrow d_{min} = V\sqrt{2T_b}$$

B. (7 points) A manufacturer of resistors has two factories and the resistors are guaranteed to have a resistance value within 2Ω of the nominal value.

- Under hypothesis 0, resistors are manufactured at factory 0 and have a variation around the nominal value that is a random variable X which is uniformly distributed in the interval $(-2,2)$ and
- Under hypothesis 1, resistors are manufactured at factory 1 and have a variation around the nominal value that is a random variable X with pdf given by $f_X(u) = \frac{1}{4}(2 - |u|)$ for $u \in (-2,2)$ and zero elsewhere.

a) (4 points) State the ML decision rule in terms of a threshold test on the observed value of $|X|$.

Let $Y = |X|$. Let the pdf of Y under H_0 denoted by f_0 and under H_1 be denoted by f_1. Then,

$f_0(y) = \frac{1}{2} \quad 0 \leq y \leq 2$, and $f_1(y) = 1 - \frac{y}{2} \quad 0 \leq y \leq 2$

Likelihood ratio, $\Lambda(Y) = \frac{1 - Y/2}{1/2} = 2 - Y$

For ML, $H_1 : \Lambda(Y) > 1 \Rightarrow Y < 1$

$\Rightarrow H_1 : |X| < 1$

$H_0 :$ ~~yes~~ $Y \geq 1$ or $|X| \geq 1$

[∵ for ML: a priori ~~fun~~ information is uniform.
i.e. $P_H(H_0) = P_H(H_1) = 1/2$.]

b) (3 points) State the MAP decision rule in terms of a threshold test on the observed value of $|X|$.

$H_1 : \Lambda(Y) > \frac{P_H(H_0)}{P_H(H_1)} \Rightarrow Y < 2 - \frac{P_H(H_0)}{P_H(H_1)}$