# UCLA — Electrical Engineering Dept.
# EE132A: Communication Systems
# Final Exam
# Solutions

1. *Let* $u(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)\mathrm{d}\tau\right)$ *be an FM-modulated signal, with* $f_c =$ *500 kHz,* $k_f = 100$ *Hz/V, and* $m(t) = a\cos(2\pi f_m t)$*, where* $a = 110$ *V and* $f_m = 2$ *kHz. The channel adds Gaussian white noise with power spectral density* $N_0/2$*, with* $N_0 = 10^{-8}$ *W/Hz, and attenuates the signal power by 40 dB.*

   (a) *Write* $u(t)$ *as a summation of sinusoids of the type* $\cos(2\pi(f_c + nf_m)t)$*, for* $n = 0, \pm1, \pm2, \ldots$
   From (4.2.5), $u(t) = \sum_{-\infty}^{\infty} A_c J_n(\beta_f)\cos(2\pi(f_c + nf_m)t)$, where $\beta_f = k_f a/f_m = 100 \cdot 110/2000 = 5.5$.

   (b) *What is the power of* $u(t)$*, in terms of* $A_c$*?*
   The power of $u(t)$ is simply $P_T = A_c^2/2$.

   (c) *What is the power of the component at* $f = f_c$*?*
   The component at $f = f_c$ is (from the first part) $A_c J_0(\beta_f)\cos(2\pi f_c t)$ and its power is $A_c^2 J_0(\beta_f)/2 \approx 0$ (from the Bessel function table provided).

   (d) *What is the bandwidth of* $u(t)$*?*
   From Carson's rule, $B_c = 2(\beta_f + 1)W = 2 \cdot 6.5 \cdot 2000 = 26$ kHz. (Note that $W = f_m = 2$ kHz.)

   (e) With no preemphasis/deemphasis, *compute the ratio between the SNR at the output of the FM demodulator and the SNR of the baseband system with the same received power.*
   From (6.2.21), $\mathrm{SNR}_o/\mathrm{SNR}_b = 3P_M\beta_f^2/\max|m(t)|^2 = 3 \cdot (a^2/2)(5.5)^2/a^2 = 45.375$ or 16.57 dB.

   (f) *Now assume preemphasis/deemphasis filtering is used, where the deemphasis filter's frequency response is given by*

$$H_d(f) = \frac{1}{1 + j\frac{f}{f_0}}, \quad f_0 = 1 \text{ kHz.}$$

   *If the SNR at the output of the deemphasis filter is 20 dB, what is the carrier amplitude,* $A_c$*?*
   From (6.2.42),

$$\frac{\mathrm{SNR}_{PD}}{\mathrm{SNR}_o} = \frac{\left(\frac{W}{f_0}\right)^3}{3\left(\frac{W}{f_0} - \tan^{-1}\frac{W}{f_0}\right)} = \frac{8}{3(2 - \tan^{-1}(2))} \approx 3,$$

   or 4.75 dB. From the previous part,

$$\mathrm{SNR}_{b,\mathrm{dB}} = \mathrm{SNR}_{o,\mathrm{dB}} - 16.57 = \mathrm{SNR}_{PD,\mathrm{dB}} - 4.75 - 16.57 = 20 - 4.75 - 16.57 = -1.32 \text{ dB},$$

   or 0.74. Because

$$\mathrm{SNR}_b = \frac{P_R}{N_0 W} = \frac{P_T \cdot 10^{-4}}{(10^{-8}/2) \cdot 2000} = 0.74,$$

$P_T = A_c^2/2 = 0.074$ and $A_c \approx 0.384$ V.

2. *Let binary symbols be transmitted with a 2-PAM system, where bit "0" is assigned to signal $s_1(t)$ and bit "1" is assigned to signal $s_2(t) = -s_1(t)$, where*

$$s_1(t) = \begin{cases} A, & 0 < t < T, \\ 0, & \textit{elsewhere.} \end{cases}$$

*Let the probability to send a "0" be three times as high as the probability to send a "1." Let $A = 1$ V, $T = 0.05$ ms. The additive white Gaussian noise has spectral density $N_0/2$, with $N_0 = 10^{-4}$ W/Hz.*

   (a) *Draw the diagram of the optimal correlation receiver.*
   Block diagram where $r(t)$ is multiplied by $\psi_1(t)$, then goes through an integrator and a sampler at $t = T$. The output is compared to a threshold $\alpha$. The basis function $\psi_1(t)$ is given by

$$\psi_1(t) = \begin{cases} \sqrt{\frac{1}{T}}, & 0 < t < T, \\ 0, & \text{elsewhere.} \end{cases}$$

   (b) *What is the optimal threshold, $\alpha^*$?*
   From (8.3.51), with $\mathcal{E}_b = A^2 T = 5 \cdot 10^{-5}$ J,

$$\alpha^* = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{P(s_2)}{P(s_1)} = \frac{10^{-4}}{4\sqrt{5 \cdot 10^{-5}}} \ln(1/3) \approx -4 \cdot 10^{-3}.$$

   (c) *Compute the probability of bit error. (Use the approximation $\mathrm{Q}(x) \approx \frac{1}{2}e^{-x^2/2}$.)*

$$\begin{aligned}\mathbb{P}(e) &= 0.25\mathrm{Q}\left(\frac{\sqrt{\mathcal{E}_b} + \alpha^*}{\sqrt{N_0/2}}\right) + 0.75\mathrm{Q}\left(\frac{\sqrt{\mathcal{E}_b} - \alpha^*}{\sqrt{N_0/2}}\right) \\ &= 0.25\mathrm{Q}\left(\frac{\sqrt{5 \cdot 10^{-5}} - 4 \cdot 10^{-3}}{\sqrt{10^{-4}/2}}\right) + 0.75\mathrm{Q}\left(\frac{\sqrt{5 \cdot 10^{-5}} + 4 \cdot 10^{-3}}{\sqrt{10^{-4}/2}}\right) \approx 0.127,\end{aligned}$$

   or, using the approximation $\mathrm{Q}(x) \approx \frac{1}{2}e^{-x^2/2}$, $\mathbb{P}(e) \approx 0.226$. (Note that the approximation is not very good here because the argument of $\mathrm{Q}(\cdot)$ is not sufficiently large.)

   (d) *If the receiver has mistakenly set the threshold at $\alpha = 0$, what is the probability of bit error? (Use the same numerical values and approximation as in the previous part.)*
   Same as before, with $\alpha^*$ replaced by 0, so

$$\mathbb{P}(e) = 0.25\mathrm{Q}\left(\frac{\sqrt{0.05}}{\sqrt{10^{-4}/2}}\right) + 0.75\mathrm{Q}\left(\frac{\sqrt{0.05}}{\sqrt{10^{-4}/2}}\right) \approx 0.159,$$

   or using the approximation for $\mathrm{Q}(\cdot)$, $\mathbb{P}(e) \approx 0.3$.

3. *Let the basis functions of a quaternary digital baseband communication system be given by*

$$\psi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 < t < \frac{T}{2}, \\ 0, & \textit{elsewhere}, \end{cases} \quad \psi_2(t) = \begin{cases} \sqrt{\frac{2}{T}}, & \frac{T}{2} < t < T, \\ 0, & \textit{elsewhere}, \end{cases}$$

*where $T = 2$ ms. Let the noise be white and Gaussian with power spectral density $N_0/2 = 10^{-2}$ W/Hz.*

(a) *Draw the waveforms of the four equiprobable signals, which are represented in the $(\psi_1, \psi_2)$ plane as points $(a, a)$, $(a, -a)$, $(-a, a)$, and $(-a, -a)$, with $a = 0.1$ V.*
Signal $s_1(t) = a\psi_1(t) + a\psi_2(t)$ is constant equal to $a\sqrt{2/T}$ from 0 to $T$, signal $s_2(t) = a\psi_1(t) - a\psi_2(t)$ is equal to $a\sqrt{2/T}$ from 0 to $T/2$ and $-a\sqrt{2/T}$ from $T/2$ to $T$, etc.

(b) *Draw the diagram of the optimal matched filter receiver.*
Block diagram with $r(t)$ splitting into two branches, one going into a linear filter with impulse response $\psi_1(T - t)$ and the other into a filter with impulse response $\psi_2(T-t)$; the outputs are the coordinates of a point in the $(\psi_1, \psi_2)$ plane and the main quadrants are the decision regions.

(c) *What are the average bit energy and the average bit power?*
$\mathcal{E}_s = a^2(2/T)T = 2a^2 = 2\mathcal{E}_b$, $T_b = T/2$, so $\mathcal{E}_b = a^2 = 10^{-2}$ J, and $P_b = \mathcal{E}_b/T_b = 2a^2/T = 10$ W.

(d) *Compute the conditional symbol error probability $\mathbb{P}(e|\mathbf{s}_1)$. (It is probably easier to compute the probability of correct reception $\mathbb{P}(c|\mathbf{s}_1)$.)*
Because the noise components, $n_1$ and $n_2$, are independent Gaussian RVs with zero mean and variance $N_0/2$,

$$\mathbb{P}(e|\mathbf{s}_1) = 1 - \mathbf{P}(c|\mathbf{s}_1) = 1 - \left(1 - \mathrm{Q}\left(\sqrt{\frac{\mathcal{E}_b}{N_0/2}}\right)\right)^2 = 1 - (1 - \mathrm{Q}(1))^2 \approx 0.29,$$

using Table 5.1, or, using the approximation for $\mathrm{Q}(\cdot)$, $\mathbb{P}(e|\mathrm{s}_1) \approx 0.51$, again, not a very good approximation.

(e) *Compute the exact probability of symbol error.*
Because all the conditional probabilities are equal and the symbols are equiprobable, $\mathbb{P}(e) = \mathbb{P}(e|\mathbf{s}_1) \approx 0.29$ (or 0.51 using the approximation).

(f) *Assuming now that $a$ is to be determined, compute the union bound of the probability of symbol error in terms of $a$.*
The minimum distance between any two constellation points is $d_{\min} = 2a$, hence the union bound is $\mathbb{P}(e) \approx 3\mathrm{Q}((d_{\min}/2)/\sqrt{N_0/2}) = 3\mathrm{Q}(a/\sqrt{N_0/2}) = 3\mathrm{Q}(10a)$. It is also acceptable to use $\mathbb{P}(e) \approx 2\mathrm{Q}(10a)$, because there are two points at minimum distance for each of the four constellation points, or just $\mathbb{P}(e) \approx \mathrm{Q}(10a)$.

(g) *If a symbol error probability of* $10^{-4}$ *is desired, what should the value of* $a$ *be? (Use the union bound result obtained in the previous part.)*
From the previous point, $a = 0.1\text{Q}^{-1}(10^{-4}/3) \approx 0.4$. (using the other approximations, $a = 0.1\text{Q}^{-1}(10^{-4}/2) = 0.39$, $a = 0.1\text{Q}^{-1}(10^{-4}) = 0.37$.)

4. *An* $M$*-PAM signal is sent over a bandlimited channel with frequency response* $C(f) = \Lambda(f/W)$*, with* $W = 30$ *kHz.*

   (a) *If a square-root raised cosine pulse,* $g_T(t)$*, is used at the transmitter with* 50% *roll-off, what is the highest symbol rate the channel can support?*
   If the bandwidth of the raised cosine is to be contained in the channel, $\frac{1+\alpha}{2T} \leq W$, therefore $R_s = 1/T \leq 2W/1.5 = 60000/1.5 = 40$ ksps (symbols per second).

   (b) *Can a 4-PAM be used to transmit at a rate of* 100 *kbps, and if not, what roll-off factor should be used instead?*
   Because $R_b = kR_s = 2R_s$, with $\alpha = 0.5$, the channel can only support up to 80 kbps. In order to have $R_b = 100$ kbps, we need $R_b = 4W/(1+\alpha)$, i.e., $1 + \alpha = 1.2$, or $\alpha = 20\%$.

   (c) *What is the frequency response of the receiver filter,* $G_R(f)$*?*
   Because $G_T(f) = \sqrt{X_{rc}(f)}$ (see the first part), and $G_T(f)C(f)G_R(f) = X_{rc}(f)$,

$$G_R(f) = \frac{\sqrt{X_{rc}(f)}}{\Lambda(f/W)}, \quad |f| < W.$$