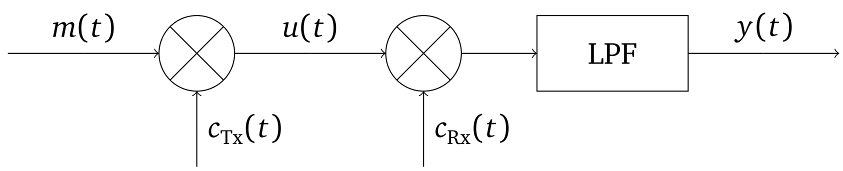
Midterm Exam - Solutions

1. A DSB-SC modulated signal, $u(t)$, is demodulated by applying it to a *coherent detector*. Assume the carrier used at the transmitter is given by $c_{\mathrm{Tx}}(t) = A_c \cos(2\pi f_c t)$, while the carrier used by the receiver has a frequency error Δf, i.e., $c_{\mathrm{Rx}} = \cos(2\pi(f_c + \Delta f)t)$.



The cutoff frequency of the LPF is W Hz.

(a) (5 points) Evaluate the effect of the frequency error, Δf, in the demodulator's local carrier, i.e., write the expression of $y(t)$, the output of the low-pass filter, assuming that $\Delta f \ll f_c$ and $\Delta f \ll W$, where W is the bandwidth of the message signal, $m(t)$.

Solution: The received signal is $r(t) = u(t) = A_c m(t)\cos(2\pi f_c t)$. Thus,

$$\begin{aligned} r(t)\cos(2\pi(f_c + \Delta f)t) &= A_c m(t) m(t)\cos(2\pi f_c t)\cos(2\pi(f_c + \Delta f)t) \\ &= \frac{A_c m(t)}{2}\left(\cos(2\pi \Delta f\, t) + \cos(2\pi(2f_c + \Delta f)t)\right). \end{aligned}$$

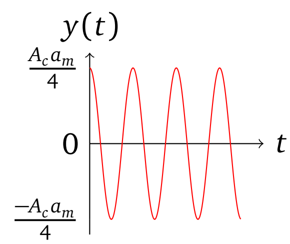
The output signal is $y(t) = \frac{A_c m(t)}{2}\cos(2\pi\Delta f\, t) \star 2W\mathrm{sinc}(2Wt)$, where $\mathcal{F}^{-1}[\Pi(\frac{f}{2W})] = 2W\mathrm{sinc}(2Wt)$.

(b) (5 points) Write the expression of $Y(f)$ in terms of $M(f)$.

Solution: $Y(f) = \frac{A_c}{4}\left(M(f - \Delta f) + M(f + \Delta f)\right)\Pi(\frac{f}{2W})$, for $|f| \le W$.

(c) (5 points) Assume $m(t) = a_m \cos(2\pi f_m t)$. Illustrate your answer by writing the expression of $y(t)$ and sketching the demodulated signal. (Note that $f_m = W$.)

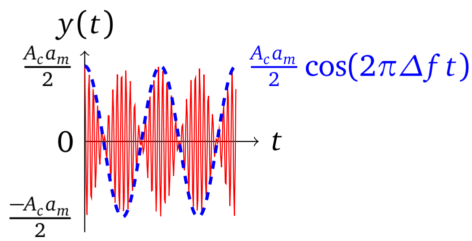
Solution: After the lowpass filter with bandwidth $W = f_m$ (not including $f_m + \Delta f$), $y(t) = \frac{A_c a_m}{4}\cos(2\pi(f_m - \Delta f)t)$ Drawing of $y(t)$ is given as:



Solution: After the lowpass filter with bandwidth $W = f_m$ (including $f_m + \Delta f$),

$$y(t) = \frac{A_c a_m}{2}\cos(2\pi f_m t)\cos(2\pi \Delta f\, t).$$

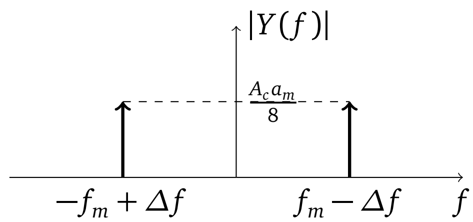
Drawing of $y(t)$ is given as:



(d) (5 points) Still assuming $m(t) = a_m\cos(2\pi f_m t)$, plot the amplitude spectrum of $y(t)$.

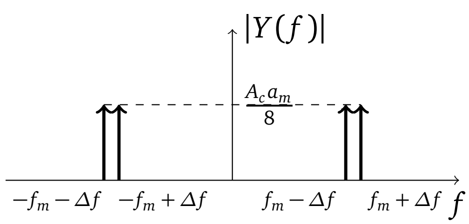
Solution: $M(f) = \frac{a_m}{2}(\delta(f - f_m) + \delta(f + f_m))$. When the frequency components of $f_m + \Delta f$ and $-f_m - \Delta f$ are not included, we obtain

$$Y(f) = \frac{A_c a_m}{8}(\delta(f - \Delta f + f_m) + \delta(f + \Delta f - f_m)).$$



Solution: $M(f) = \frac{a_m}{2}(\delta(f - f_m) + \delta(f + f_m))$. When the frequency components of $f_m + \Delta f$ and $-f_m - \Delta f$ are included, we obtain

$$Y(f) = \frac{A_c a_m}{8}(\delta(f - \Delta f + f_m) + \delta(f + \Delta f - f_m) + \delta(f + \Delta f + f_m) + \delta(-f - \Delta f - f_m)).$$



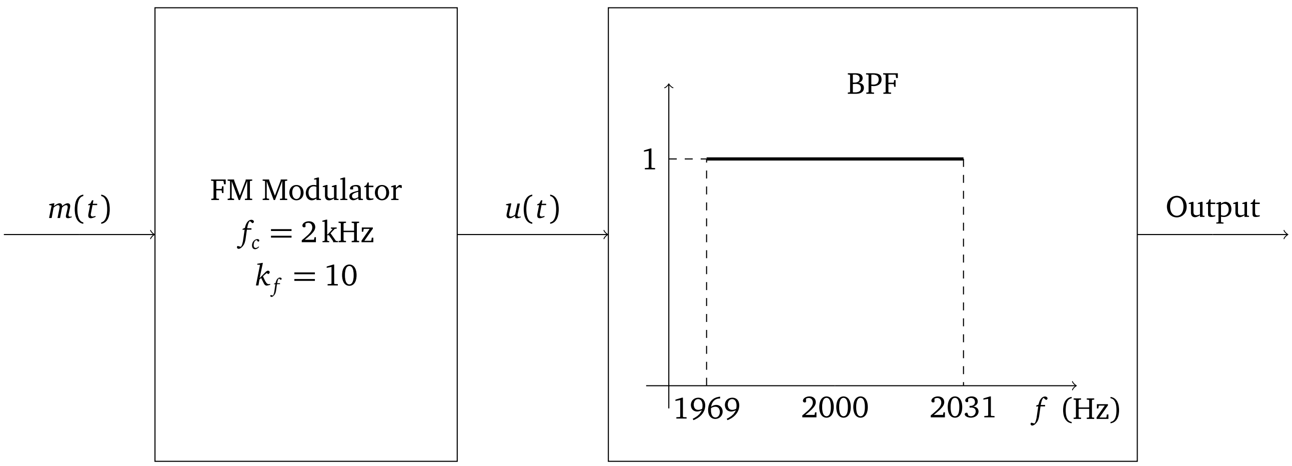
2. (20 points) The modulating signal that is input to an FM modulator is given by

$$m(t) = 10\cos(16\pi t), \quad \text{for } t \geq 0.$$

The output of the FM modulator is

$$u(t) = 10\cos\left(4000\pi t + 2\pi k_f \int_{-\infty}^{t} m(\tau)\,\mathrm{d}\tau\right),$$

where $k_f = 10$, see figure below.



If the output of the modulator, $u(t)$ is passed through an ideal BPF centered at f_c and with a bandwidth of 62 Hz, determine the power of the frequency components at the output of the filter. Please compute the percentage of the power of $u(t)$ that appears at the output of the BPF?

Solution: $\int_{-\infty}^{t} m(\tau)\,\mathrm{d}\tau = \frac{10}{16\pi}\sin(16\pi t)$. We denote $\beta = 2\pi k_f \frac{10}{16\pi} = 10\frac{10}{8} = 12.5$, $f_c = 2000$ [Hz], $A_c = 10$ [V] and $f_m = 8$ [Hz]. Thus, $u(t) = A_c\cos\left(2\pi f_c t + \beta\sin(2\pi f_m t)\right)$. Then, $u(t)$ can be expressed as:

$$u(t) = \sum_{n=-\infty}^{\infty} A_c \mathrm{J}_n(\beta)\cos(2\pi(f_c + nf_m)t).$$

The power of the frequency components at the output of the filter is

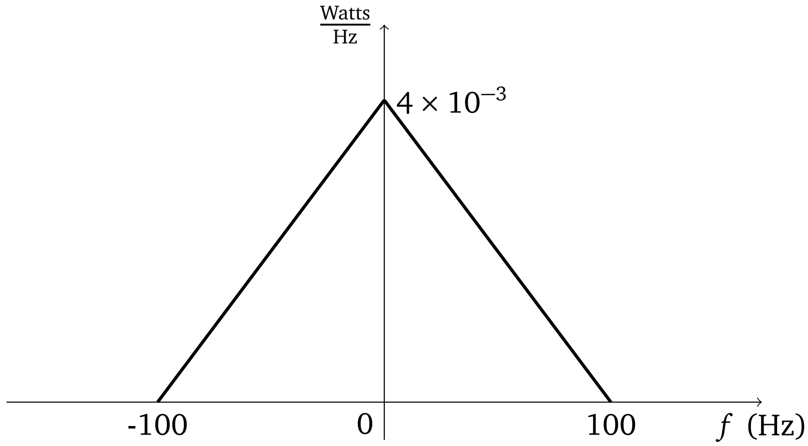
$$\frac{A_c^2}{2}\sum_{n=-3}^{3}\mathrm{J}_n^2(\beta) = 8.35 \text{ [Watts]},$$

or 3.5 [Watts] (if only use fist decimal digit). The total power of $u(t)$ is $\frac{A_c^2}{2} = 50$ [Watts]. As a result, the percentage of the power of $u(t)$ that appears at the output of the BPF is given as:

$$\sum_{n=-3}^{3}\mathrm{J}_n(\beta)^2 = \mathrm{J}_0(\beta)^2 + 2\times\sum_{n=1}^{3}\mathrm{J}_n(\beta)^2 = 0.1607 = 16.7\%,$$

or 7% (if only use first decimal digit and truncate the second decimal digit).

3. A message signal, $m(t)$, such that $-2 \le m(t) \le 2$ volts has power spectral density given by



The white Gaussian noise on the channel has a power spectral density $\frac{N_0}{2}$, where $N_0 = 1 \times 10^{-6} \frac{\text{W}}{\text{Hz}}$.

(a) (5 points) Assuming that the message modulates a carrier of amplitude $A_c = 1\,\text{V}$ in a DSB-SC scheme, what is the signal-to-noise ratio at the output of the coherent demodulator? (Assume that the receiver is perfectly synchronized to the transmitter.)

Solution: The output $y(t) = \frac{1}{2}A_c m(t) + \frac{n_c(t)}{2}$. The signal power is $P_{o,s} = \frac{A_c^2}{4}P_M = 0.25 \times 0.4 = 0.1$ [watts]. The noise power is $P_{o,n} = \frac{1}{4}2N_0 W = 0.25 \times 2 \times 10^{-6} \times 100 = 5 \times 10^{-5}$ [watts]. Thus, the corresponding SNR is $\frac{0.1}{5\times10^{-5}} = 2000$.

(b) (5 points) If $m(t)$ modulates the same carrier in a conventional AM scheme with a modulation index $= 0.9$, what is the signal-to-noise ratio at the output of the envelope detector, after the DC has been removed? (Can you make the assumption that the received signal-to-noise ratio is high?)

Solution: After removing the DC component, we obtain the output of the envelope detector is $y(t) = A_c a m_n(t) + n_c(t)$. Since $|m(t)| \le 2$, then $m_n(t) = \frac{m(t)}{2}$. The power content of $m_n(t)$ is $P_{m_n} = \frac{P_m}{4} = 0.1$ [Watts]. The signal power is $P_{o,s} = A_c^2 a^2 P_{M_n}$ [Watts]. The noise power is $P_{o,n} = 2WN_0$ [watts]. Since modulation index is $a = 0.9$, the corresponding SNR $= \frac{A_c^2 a^2 P_{M_n}}{2WN_0} = \frac{0.9^2 \times \frac{P_M}{4}}{2\times10^{-4}} = 405$. Thus, we can assume that the received signal-to-noise ratio is high.

(c) (5 points) Still assuming a conventional AM scheme with a modulation index $= 0.90$, what is the signal-to-noise ratio at the output of the coherent detector, after the DC has been removed?

Solution: $y(t) = \frac{A_c a m_n(t)}{2} + \frac{n_c(t)}{2}$. The output SNR $= \frac{\frac{1}{4}A_c^2 a^2 P_{M_n}}{\frac{1}{4}2WN_0} = \frac{0.9^2\times0.1}{2\times10^{-4}} = 405$.

(d) (5 points) Which of the two modulation schemes, DSB-SC or conventional AM, guarantees a higher signal-to-noise ratio at the output of the corresponding demodulator?

> **Solution:** We observe that DSB-SC AM can guarantee a higher signal-to-noise ratio at the output of the corresponding demodulator.