EE 132A
Introduction to Communication Systems
Instructor: Lara Dolecek

Spring 2016 Midterm
Wednesday, April 27, 2016

Maximum score is 100 points. You have 110 minutes to complete the exam. Please show your work. Good luck!

Your Name: Kai-Zhou Yang

Your ID Number: 304343286

Name of person on your left: N/A

Name of person on your right: Mignon Huang

Excellent !!

Problem	Score	Possible
1	20	20
2	20	20
3	20	20
4	20	20
5	17	20
Total	97	100

20

1. (20 pts) Let X and Y be independent continuous random variables, with X uniform on $[0, 1]$ and Y uniform on $[0, 2]$. Let $Z(t)$ be a random process defined as

$$Z(t) = X \cos 2\pi ft + Y \sin 2\pi ft.$$

(a) Compute mean and autocorrelation of $Z(t)$.

(b) Is $Z(t)$ strict sense stationary ? Provide an argument.

(c) Is $Z(t)$ wide sense stationary ? Provide an argument.

a) $\mu(t) = E[Z(t)] = E[X]\cos 2\pi ft + E[Y]\sin 2\pi ft$ $* \; E[X] = \frac{1}{2}; \; E[Y] = 1$

$= \boxed{\frac{1}{2}\cos 2\pi ft + \sin 2\pi ft}$

$R(t_1,t_2) = E[Z(t_1)Z(t_2)] = E[X^2]\cos 2\pi f t_1 \cos 2\pi f t_2 + E[XY]\cos 2\pi f t_1 \sin 2\pi f t_2$

$+ E[XY]\sin 2\pi f t_1 \cos 2\pi f t_2 + E[Y^2]\sin 2\pi f t_1 \sin 2\pi f t_2$

$* \; E[X^2] = \frac{1}{3}; \; E[Y^2] = \int_0^2 x\left(\frac{1}{2}\right)dx = \frac{1}{2}\left(\frac{1}{3}x^3\Big|_0^2\right) = \frac{1}{2}\cdot\frac{8}{3} = \frac{4}{3}$

$* \; E[XY] = E[X]E[Y] = \frac{1}{2}$

$\Rightarrow R(t_1,t_2) = \boxed{\frac{1}{3}\left[\cos(2\pi f(t_1-t_2))\right] + \sin 2\pi f t_1 \sin 2\pi f t_2 + \frac{1}{2}\sin 2\pi f(t_1+t_2)}$

b) No. It isn't even W.S.S.

c) No. $\mu(t) \neq \mu \in \mathbb{R}$

$\frac{20}{20}$

2. (20 pts) Suppose a strict sense stationary random process $X(t)$ has power spectral density $S_X(f)$.

   (a) What is the power spectral density of the random process $Y(t)$ given by $Y(t) = X(t) - X(t-T)$, where T is an arbitrary positive constant.

   (b) Is the process $Y(t)$ wide-sense stationary ? Justify your answer.

a) $R_Y(\tau) = E[(X(t+\tau) - X(t+\tau-T))(X(t) - X(t-T))]$

$= E[X(t+\tau)X(t)] - E[X(t+\tau)X(t-T)] - E[X(t+\tau-T)X(t)] + E[X(t+\tau-T)X(t-T)]$

$= R_X(\tau) - R_X(\tau+T) - R_X(\tau-T) + R_X(\tau) = 2R_X(\tau) - R_X(\tau+T) - R_X(\tau-T)$

$\Rightarrow S_Y(f) = 2S_X(f) - (e^{+j2\pi f T} S_X(f) + e^{-j2\pi f T} S_X(f))$ ★ $X(t)$ real-valued $\Rightarrow R_X(\tau)$ real, $R_X(\tau)$ even $\Rightarrow S_X(f)$ real & even

$= \boxed{2S_X(f) - 2S_X(f)\cos 2\pi f T}$ ✓

b) $\mu_Y(t) = E[X(t) - X(t-T)] = E[X(t)] - E[X(t-T)] = \cancel{2\mu_X}\ 0 \Rightarrow$ constant $\mu_Y(t) = \mu_Y \in \mathbb{R}$

$\boxed{\text{Yes.}}$ ✓ $Y(t)$ is WSS since $\mu_Y(t) = \mu_Y$ & $R_Y(t_1,t_2) = R_Y(\tau)$ ✓

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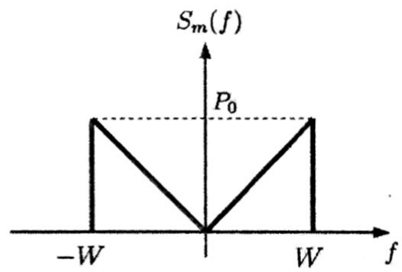
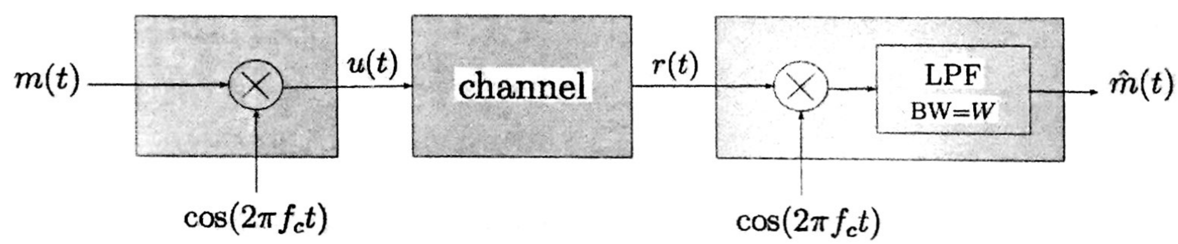
3. (20 pts) Baseband signal $m(t)$ has the power spectral density as shown in figure 1(a). Signal $m(t)$ passes through a DSB amplitude modulator. The signal at the out put of the modulator is named as $u(t)$. Then, $u(t)$ is fed to a channel with additive noise that has a power spectral density $N_0/2$ within the passband of the signal. Name the output of the channel as $r(t)$. Signal $r(t)$ passes through a demodulator followed by an ideal low-pass filter with bandwidth W and signal $\hat{m}(t)$ is recovered at the output of LPF.

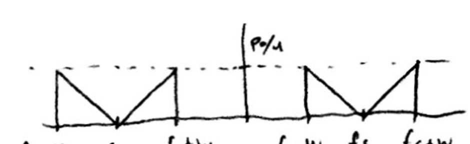
(a) Sketch the power spectral density of signals $u(t)$, $r(t)$, and $\hat{m}(t)$.





$u(t)$ cyclostationary w/ $\bar{R}_u(\tau) = \frac{1}{2} R_m(\tau)\cos(2\pi f_c \tau) \Rightarrow S_u(f) = \frac{1}{2} S_m(f) * \left(\frac{1}{2}\delta(f-f_c) + \frac{1}{2}\delta(f+f_c)\right)$

$= \frac{1}{4}\left[S_m(f+f_c) + S_m(f-f_c)\right]$



+7

A check: $R_u(t_1,t_2) = E\left[\cos(2\pi f_c t_1) M(t_1) M(t_2) \cos(2\pi f_c t_2)\right]$, $\tau = t_1 - t_2$

$= R_M(\tau)\left[\cos(2\pi f_c(t_1+t_2)) + \cos(2\pi f_c \tau)\right]\frac{1}{2}$

$= \frac{1}{2} R_M(\tau)\left[\cos(2\pi f_c \tau) + \cos(2\pi f_c(2t_1+\tau))\right]$ periodic w/ $T_0 = \frac{1}{2f_c}$

$\Rightarrow \bar{R}_u(\tau) = \frac{1}{2} R_M(\tau)\cos(2\pi f_c \tau) + \frac{1}{2} R_M(\tau)\left[\frac{1}{T_0}\int_0^{T_0} \cos(4\pi f_c t + 2\pi f_c \tau)\,dt\right] = \frac{1}{2} R_M(\tau)\cos(2\pi f_c \tau)$ ✓

$\left\{ \sin(4\pi f_c t + 2\pi f_c \tau)\Big|_0^{1/2f_c} = 0 \right\}$

Extra work space.

$r(t) = u(t) + n(t)$

$\Rightarrow R_R(t,\tau) = E\left[(U(t+\tau) + N(t+\tau))(U(t) + N(t))\right]$

$= R_U(t,\tau) + R_N(\tau) + R_{U,N}(\tau) + R_{U,N}(-\tau)$

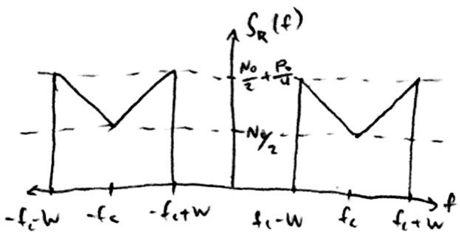
* Realistically, U & N are independent & therefore uncorrelated

$\Rightarrow R_R(t,\tau) = R_U(t,\tau) + R_N(\tau) + \mu_U(t+\tau)\mu_N(t) + \mu_U(t)\mu_N(t+\tau)$

* Assuming $W < f_c$, $S_N(0) = 0 \Rightarrow E[N] = 0 \Rightarrow \mu_N = 0 \ \forall t$

$\Rightarrow R_R(t,\tau) = R_U(t,\tau) + R_N(\tau) \Rightarrow \overline{R_R}(\tau) = \overline{R_U}(\tau) + R_N(\tau)$

$\Rightarrow S_R(f) = S_U(f) + S_N(f)$



+6

$\hat{m}(t) = LPF\{r(t)\cos(2\pi f_c t)\} = m(t)\cos^2(2\pi f_c t) + (n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t))\cos(2\pi f_c t)$

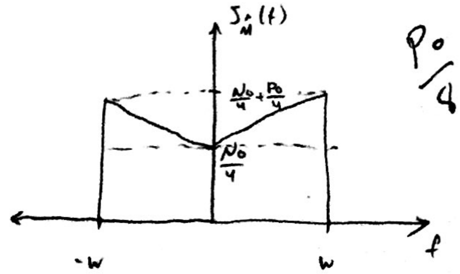
$= m(t)\left(\frac{1}{2} + \underbrace{\cancel{\tfrac{1}{2}\cos(4\pi f_c t)}}_{LPF}\right) + n_c(t)\left(\frac{1}{2} + \underbrace{\cancel{\tfrac{1}{2}\cos(4\pi f_c t)}}_{LPF}\right) - n_s(t)\left(\underbrace{\cancel{\tfrac{1}{2}\sin(4\pi f_c t)}}_{LPF} + \cancel{\tfrac{1}{2}\sin(0)}\right)$

$= (m(t) + n_c(t))\frac{1}{2}$

* Similarly, M & N_c are uncorrelated & N_c is 0 mean

$\Rightarrow S_{\hat{M}}(f) = \frac{1}{4}S_M(f) + \frac{1}{4}S_{N_c}(f)$

* $S_{N_c} = \begin{cases} N_0 & |f| < W \\ 0 & \text{o.w.} \end{cases} = \begin{cases} S_N(f-f_0) + S_N(f+f_0) & |f| < f_0 \\ 0 & \text{o.w.} \end{cases}$



$P_0/4$

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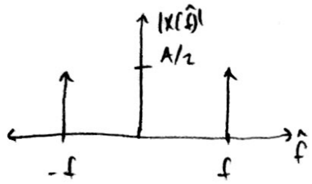
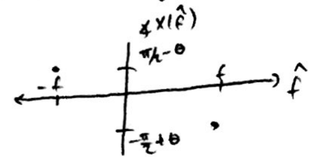
4. (20 pts) Let $x(t) = A\sin(2\pi f t + \theta)$. Sketch the frequency representation of $x(t)$ and the frequency representation of its in-phase and quadrature components, $x_c(t)$ and $x_s(t)$.

$$x(t) = A\cos\left(2\pi f t - \frac{\pi}{2} + \theta\right) = \frac{A}{2}\left[e^{j(-\frac{\pi}{2}+\theta)} e^{j2\pi f t} + e^{j(\frac{\pi}{2}-\theta)} e^{-j2\pi f t}\right]$$

$$= \frac{A}{2}\left[-je^{j\theta} e^{j2\pi f t} + je^{-j\theta} e^{-j2\pi f t}\right]$$

$$\Rightarrow X(\hat{f}) = \frac{A}{2}\left[-je^{j\theta}\delta(\hat{f}-f) + je^{-j\theta}\delta(\hat{f}+f)\right]$$





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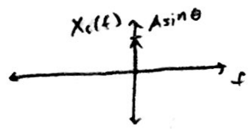
$$\Rightarrow x(t) = \mathrm{Re}\left\{A e^{j(2\pi f t - \frac{\pi}{2}+\theta)}\right\} \Rightarrow z(t) = A e^{j(-\frac{\pi}{2}+\theta)} e^{j2\pi f t} = x_l(t) e^{j2\pi f t}$$

$$\Rightarrow x_l(t) = A e^{j(-\frac{\pi}{2}+\theta)} = x_c(t) + j x_s(t) \Rightarrow \begin{cases} x_c(t) = A\cos(-\frac{\pi}{2}+\theta) = A\sin\theta \\ x_s(t) = A\sin(-\frac{\pi}{2}+\theta) = -A\cos\theta \end{cases}$$

$\star\ \mathcal{F}[1] = \delta(f)$

$\Rightarrow$



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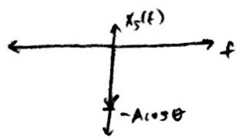
$$\Rightarrow X_c(f) = A\sin\theta\,\delta(f)$$
$$X_s(f) = -A\cos\theta\,\delta(f)$$



+7

17/20

5. (20 pts) True or False.

Circling the correct answer is worth +2 points, circling an incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) Power spectral density is the Fourier transform of the squared mean of the given random process.

TRUE FALSE

(b) If $X(t)$ is a Gaussian random process, then it is completely specified by its mean and autocorrelation functions.

TRUE FALSE

(c) For two WSS random processes $X(t)$ and $Y(t)$, $R_{X,Y}(\tau) = R_{X,Y}(-\tau)$.

TRUE FALSE

(d) Consider an LTI system described in time by a delta function. Then, power spectral density of the output is half the power spectral density of the input.

TRUE FALSE

(e) Auto correlation of a random process $X(t)$ can be negative.

TRUE FALSE

(f) Hilbert transform of $\delta(t)$ is π.

TRUE FALSE

(g) For a bandpass signal $x(t)$ of bandwidth W and centered at frequency f_0, its lowpass representation $x_\ell(t)$ does not have any frequencies larger than $f_0 + W/2$.

TRUE FALSE * Assuming $f > 0$

(h) If X, Y, and Z are jointly Gaussian random variables, then each of X, Y, and Z is a Gaussian with mean 0.

TRUE FALSE

(i) Two random processes that are independent and each is WSS, are jointly WSS.

TRUE FALSE

(j) DSB-SC AM requires half the bandwidth of SSB-SC AM.

TRUE FALSE