1. (8 pts) Suppose X is a random variable uniformly distributed on the interval $[-2a, a]$ for some $a > 0$. Compute the mean and variance of X.

$$f_X(x) = \frac{1}{a+2a} = \frac{1}{3a} \quad ; -2a \le x \le a$$

$$E[X] = \int_{-2a}^{a} \frac{x}{3a} = \left[\frac{x^2}{6a}\right]_{-2a}^{a} = \frac{a^2}{6a} - \frac{4a^2}{6a}$$

$$= -\frac{3a^2}{6a} = -\frac{a}{2}$$

$$\sigma_X^2 = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_{-2a}^{a} \frac{x^2}{3a}\, dx = \left[\frac{x^3}{9a}\right]_{-2a}^{a} = \frac{a^3}{9a} + \frac{8a^3}{9a}$$

$$= a^2$$

$$\sigma_X^2 = a^2 - \left(\frac{-a}{2}\right)^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$$

2. (4 pts) Evaluate Fourier Series and Fourier Transform of the signal $x(t) = 5\sin t + 2\cos t$.

$$T_0 = 2\pi, \quad f_0 = \frac{1}{T_0} = \frac{1}{2\pi}$$

$$x(t) = \frac{5}{2j}e^{jt} - \frac{5}{2j}e^{-jt} + \frac{2}{2}e^{jt} + \frac{2}{2}e^{-jt}$$

$$= (1 - 2.5j)\,e^{jt} + (1+2.5j)\,e^{-jt}$$

$$x_n = \begin{cases} 1-2.5j & ; \; n = 1 \\ 1+2.5j & ; \; n = -1 \\ 0 & ; \; \text{otherwise} \end{cases}$$

Fourier transform:

$$F[x(t)] = X(f)$$

$$= \frac{5}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right]$$

$$+ \; \delta(f-f_0) + \delta(f+f_0)$$

3. (8 pts) Suppose $X = (X_1, X_2)$ is a bivariate random variable.

(a) Can matrix C_1 be a covariance matrix of X for C_1 given as

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}?$$

Justify your argument.

(b) Can matrix C_2 be a covariance matrix of X for C_2 given as

$$C_2 = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}?$$

Justify your argument.

a) C_1 is symmetric.

C_1 is positive semidefinite:

$$[z_1 \quad z_2] \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1^2 + 4z_2^2 \geq 0$$

Therefore $C1$ can be a Covariance matrix

b) C_2 is asymmetric. Therefore C_2 Cannot be a covariance matrix.

4. (4+4+4 pts) Suppose $X = (X_1, X_2, X_3)$ is a jointly distributed Gaussian random variable with the mean vector $[1, 0, -1]$ and the covariance matrix

$$C = \begin{bmatrix} 4 & 0 & 0 \\ -0 & 1 & -1 \\ 0 & -1 & 16 \end{bmatrix}.$$

(a) Compute $\mathbb{E}[(X_1)^2]$.

(b) Compute $\mathbb{E}[(X_2)^3]$.

(c) Are random variables X_2 and X_3 independent ? Justify your answer.

a) $C_{11} = \sigma_{X_1}^2 = E[X_1^2] - (E[X_1])^2$

$\rightarrow \quad 4 = E[X_1^2] - 1$

$E[X_1^2] = 5$

b) X_2 is zero mean. X_2^3 is an odd function.
$f_{X_2}(x)$ is zero mean and gaussian. Therefore it is an even function. $x^3 f_{X_2}(x)$ is an ~~od~~ odd function.

$\rightarrow \quad \int_{-\infty}^{\infty} x^3 f_{X_2}(x) = 0$ (Property of odd functions).

$\rightarrow \quad E[X_2^3] = 0.$

c) $C_{23} = -1 = \mathrm{Cov}(X_2, X_3) = E[X_2 X_3] - E[X_2]E[X_3]$

$\rightarrow \quad E[X_2 X_3] \neq E[X_2]\,E[X_3]$

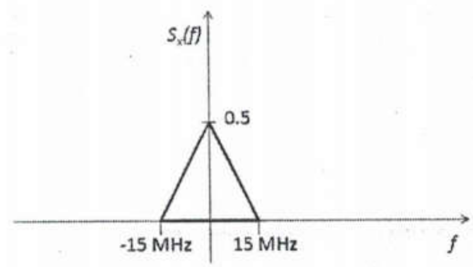
Therefore X_2 and X_3 are not independent.

5. (10+8 pts) Let $R(\tau)$ denote the autocorrelation of a WSS process $X(t)$.

(a) Prove that $R(\tau)$ is maximized at $\tau = 0$.

You may find the following useful: a) $(x-y)^2 \geq 0$, and b) Consider $Z = X(t+\tau) - X(t)$.

(b) Suppose that the power spectral density of $X(t)$ is a triangle, as shown in figure.



What is the value of $R(0)$?

a) $E[Z^2] \geq 0$

$E[(X(t+\tau) - X(t))^2] \geq 0$

$E[X^2(t+\tau) - 2X(t+\tau)X(t) + X^2(t)] \geq 0$

$E[X^2(t+\tau)] - 2E[X(t+\tau)X(t)] + E[X^2(t)] \geq 0$

$R_X(0) - 2R_X(\tau) + R_X(0) \geq 0$

$2R_X(0) \geq 2R_X(\tau)$

$R_X(0) \geq R_X(\tau)$

b) $R(0) = \int_{-\infty}^{\infty} S_X(f)\, df = \frac{1}{2} \times (30 \times 10^6) \times \frac{1}{2}$

$= 7.5 \times 10^6$

6. (5 + 10 pts) Suppose a WSS process $X(t)$ is passed through an LTI system described by $h(t) = 3/(\pi t)$. Let $Y(t)$ denote the output of this system.

   (a) What is the Fourier Transform, $H(f)$, of $h(t)$?

   (b) Suppose autocorrelation $R_{XX}(\tau)$ of $X(t)$ is given as $R_{XX}(\tau) = \cos(2\pi f_c \tau) + \sin(2\pi f_c \tau)$. Compute power spectral density of $X(t)$ and power spectral density of $Y(t)$.

a) $F[h(t)] = 3F\left[\frac{1}{\pi t}\right] = -3j\,\mathrm{sgn}(f).$

b)
$$S_x(f) = F[R_x(\tau)]$$
$$= \frac{1}{2}[\delta(f-f_c) + \delta(f+f_c)]$$
$$+ \frac{1}{2j}[\delta(f-f_c) - \delta(f+f_c)]$$

$$S_y(f) = |H(f)|^2 S_x(f)$$

$$|H(f)|^2 = |-3j\,\mathrm{sgn}(f)|^2 = (3\,\mathrm{sgn}(f))^2$$
$$= \begin{cases} 9 \; ; & f \neq 0 \\ 0 \; ; & f = 0 \end{cases}$$

$$S_y(f) = \frac{9}{2}[\delta(f-f_c)+\delta(f+f_c)] + \frac{9}{2j}[\delta(f-f_c)-\delta(f+f_c)].$$

7. (15 pts) True or False.

Circling the correct answer is worth +3 points, circling an incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) For an optimal quantizer, the quantized value for a region should be the centroid of that region.

TRUE FALSE

(b) For 2 discrete random variables X and Y, if $E[XY] \neq E[X]E[Y]$, then X and Y are not independent.

TRUE FALSE

(c) The power spectral density of white noise is a delta function.

TRUE FALSE

(d) The SNR of a DSB-SC system is higher than that of a baseband communication system.

TRUE FALSE

(e) The Hilbert transform of an even function is an odd function.

TRUE FALSE

8. (8 + 4 + 8 pts) Suppose $m(t)$ is a WSS process. Let $y(t) = m(t)\cos(2\pi f_c t)$ for some carrier frequency f_c.

(a) Show that $y(t)$ is a cyclostationary process.

(b) What kind of process is $z(t) = m(t)\cos(2\pi f_c t + \theta)$, for θ constant ? (i.e., is it strict-sense stationary, wide-sense stationary, cyclostationary, or neither ?)

(c) Now consider the process $v(t) = A \times \cos(2\pi f_c t + \theta)$, for a constant A and θ uniformly distributed on the interval $[0, 2\pi]$. What kind of process is $z(t)$? (i.e., is it strict-sense stationary, wide-sense stationary, cyclostationary, or neither ?)

a) $E[y(t)] = E[m(t)]\cos(2\pi f_c t)$

$= \mu_m \cos(2\pi f_c t)$

$\rightarrow E[y(t+T_c)] = E[y(t)]$

$R_y(t+\tau, t) = E[m(t)\cos(2\pi f_c t)\, m(t+\tau)\cos(2\pi f_c(t+\tau))]$

$= R_m(\tau)(\cos(2\pi f_c t)\cos(2\pi f_c(t+\tau)))$

$R_y(t+\tau+T_c, t+T_c) = R_m(\tau)[\cos(2\pi f_c(t+T_c)) \times \cos(2\pi f_c(t+\tau+2T_c))]$

$= R_m(\tau)[\cos(2\pi f_c t + 2\pi)\cos(2\pi f_c(t+\tau) + 4\pi)]$

$= R_m(\tau)[\cos(2\pi f_c t)\cos(2\pi f_c(t+\tau))]$

$= R_y(t+\tau, t)$.

b) $E[z(t)] = E[m(t)]\cos(2\pi f_c t + \theta)$

$= \mu_m \cos(2\pi f_c t + \theta)$

$R_z(t+\tau, t) = E[m(t)\cos(2\pi f_c t + \theta)\, m(t+\tau)\cos(2\pi f_c(t+\tau) + \theta)]$

$R_z(\tau + t + T_c, t + T_c)$

$= R_m(\tau) E[\cos(2\pi f_c t + 2\pi f_c T_c + \theta)\cos(2\pi f_c(t+\tau+2T_c)+\theta)]$

$= R_m(\tau) E[\cos(2\pi f_c t + \theta + 2\pi)\cos(2\pi f_c(t+\tau) + \theta + 4\pi)]$

$= R_m(\tau) E[\cos(2\pi f_c t + \theta)\cos(2\pi f_c(t+\tau)+\theta)]$

$= R_z(\tau + t, t).$

c) $v(t) = A\cos(2\pi f_c t + \theta)$

For any delay τ

$v(t+\tau) = A\cos(2\pi f_c t + 2\pi f_c \tau + \theta)$

$2\pi f_c \tau$ is a constant phase. $[\theta + 2\pi f_c \tau] \bmod 2\pi$ is still a uniform random variable with range $[0, 2\pi]$

$\therefore$ $v(t+\tau) = A\cos(2\pi f_c t + \theta) = v(t)$

$\therefore$ $v(t)$ is strict sense stationary.