Solutions to Practice Problem Set # 2

Problem 1

(Autocorrelation and PSD)

(a) The power spectral density of X, $S_X(f)$, is the Fourier transform of the autocorrelation, $R_X(\tau)$. The function $R_X(\tau) = e^{-2|\tau|}$ breaks down into $e^{-2|\tau|} = e^{-2\tau}u(\tau) + e^{2\tau}u(-\tau)$. The Fourier transform is thus:

$$\begin{aligned} S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau \\ &= \int_{-\infty}^{\infty} e^{-2|\tau|} e^{-j2\pi f\tau} d\tau \\ &= \int_{0}^{\infty} e^{-2\tau} e^{-j2\pi f\tau} d\tau + \int_{-\infty}^{0} e^{2\tau} e^{-j2\pi f\tau} d\tau \\ &= \frac{1}{2+j2\pi f} + \frac{1}{2-j2\pi f} \\ &= \frac{4}{4+4\pi^2 f^2} = \frac{1}{1+\pi^2 f^2} \end{aligned}$$

where the solutions to the integrals were given in the instructions as:

$$\int_{0}^{\infty} e^{-at} e^{-j2\pi ft} dt = \frac{1}{a+j2\pi f}$$

$$\int_{-\infty}^{0} e^{at} e^{-j2\pi ft} dt = \frac{1}{a-j2\pi f}$$

It is important to simplify the expression into the final form and not leave it in the form containing imaginary numbers. The Fourier transform of a real, even function is also always a real, even function. Therefore, since the autocorrelation function was real and is always even ($R_x(\tau) = R_x(-\tau)$), the PSD should also be real and even, as seen from the final expression.

(b) By the same method as section (a), we obtain

$$S_N(f) = \frac{6}{9+4\pi^2 f^2}$$

(c) For the autocorrelation of $Y(t)$, we simplify the following:

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y(t+\tau)] \\ &= E[(X(t)+N(t))(X(t+\tau)+N(t+\tau))] \end{aligned}$$

$$
\begin{aligned}
&= E[X(t)X(t+\tau) + X(t)N(t+\tau) + N(t)X(t+\tau) + N(t)N(t+\tau)] \\
&= E[X(t)X(t+\tau)] + E[X(t)N(t+\tau)] + E[N(t)X(t+\tau)] + E[N(t)N(t+\tau)] \\
&= E[X(t)X(t+\tau)] + E[X(t)]E[N(t+\tau)] + E[N(t)]E[X(t+\tau)] + E[N(t)N(t+\tau)] \\
&\quad \text{(by independence of X and N)} \\
&= E[X(t)X(t+\tau)] + E[N(t)N(t+\tau)] \\
&\quad \text{(because X and N are zero-mean)} \\
&= R_X(\tau) + R_N(\tau) = e^{-2|\tau|} + e^{-3|\tau|}
\end{aligned}
$$

The independence and zero-mean properties should be specified. Independence makes the expectation of the products equal to the product of the expectations, ie. $E[A \cdot B] = E[A]E[B]$. Because X and N also have zero mean, this means $E[X(t)] = E[N(t)] = 0$, which cancels out the cross-terms in the multiplication, leaving just the autocorrelation of X plus the autocorrelation of N.

(d) Again, the power spectral density is the Fourier transform of the autocorrelation. By linearity of the Fourier transform,

$$
\begin{aligned}
S_Y(f) &= \mathcal{F}\{R_Y(\tau)\} \\
&= \mathcal{F}\{R_X(\tau) + R_N(\tau)\} \\
&= \mathcal{F}\{R_X(\tau)\} + \mathcal{F}\{R_N(\tau)\} \\
&= S_X(f) + S_N(f) \\
&= \frac{4}{4 + 4\pi^2 f^2} + \frac{6}{9 + 4\pi^2 f^2}
\end{aligned}
$$

Problem 2

(Short questions)

(a) The total probability of error will not change. When the decision rule does not change, the conditional error probabilities $P(E|H_0)$ and $P(E|H_1)$ do not change. The total probability of error

$$P_e = p_0 P(E|H_0) + (1 - p_0)P(E|H_1) = (p_0 + 1 - p_0)P(E|H_0) = P(E|H_0) = P(E|H_1)$$

(b) No. This is because the error probability is only determined by the distance between the two points:

$$
\begin{aligned}
d &= \|\psi_0(t) - \psi_1(t)\| \\
&= \sqrt{\|\psi_0(t)\|^2 + \|\psi_1(t)\|^2}
\end{aligned}
$$

which is independent of the waveforms. Instead, it only depends on their energies.

(c) No. The Viterbi decoder finds the path $(X_1, \ldots, X_N)$ that maximizes

$$P(X|Y) = P(X_i|Y)P(\{X_j\}_{j \neq i}|Y, X_i)$$

However, it does not necessarily maximizes the individual $P(X_i|Y)$.

(d) No. For the actual channel that has $L = 1$, we only need to have a cyclic prefix that is at least 1. It also works with a cyclic prefix that is longer than that, say for example 2.

Problem 3

(Short questions)

(a) No. This constellation is not centered around origin. Let $\boldsymbol{v} = \begin{bmatrix} \sqrt{\frac{E}{2}} \\ \sqrt{\frac{E}{2}} \end{bmatrix}$. Then the one that has the lowest power for the same probability of error is given by $\tilde{\boldsymbol{s}}_0 = \boldsymbol{s}_0 - \boldsymbol{v}$ and $\tilde{\boldsymbol{s}}_1 = \boldsymbol{s}_1 - \boldsymbol{v}$.

(b) Yes. If $s_0(t)$ and $s_1(t)$ are linearly dependent, then the dimension of the space spanned by $s_0(t)$ and $s_1(t)$ is one. The orthonormal basis for $\{s_0(t), s_1(t)\}$ is either $\phi(t) = s_0(t)/\|s_0(t)\|$ or $\phi(t) = s_1(t)/\|s_1(t)\|$. Since $\int R(t)\phi(t)dt$ is the sufficient statistics, either $\int R(t)s_0(t)dt$ or $\int R(t)s_1(t)dt$ is also the sufficient statistics because $s_0(t)$ and $s_1(t)$ are just scaled versions of $\phi(t)$. On the other hand, if $s_0(t)$ and $s_1(t)$ are linearly independent, then the dimension of the space spanned by $s_0(t)$ and $s_1(t)$ is two. Let $\phi_1(t)$ and $\phi_2(t)$ be the orthonormal basis functions of $\{s_0(t), s_1(t)\}$ so that

$$s_0(t) = s_{01}\phi_1(t) + s_{02}\phi_2(t)$$
$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t)$$

The vector representations $\boldsymbol{s}_0 = \begin{bmatrix} s_{01} \\ s_{02} \end{bmatrix}$ and $\boldsymbol{s}_1 = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}$ are linearly indepdendent as well, which means we can solve $\int R(t)\phi_1(t)dt$ and $\int R(t)\phi_2(t)dt$ uniquely from the following equations:

$$\int R(t)s_0(t)dt = s_{01}\int R(t)\phi_1(t)dt + s_{02}\int R(t)\phi_2(t)dt$$
$$\int R(t)s_1(t)dt = s_{11}\int R(t)\phi_1(t)dt + s_{12}\int R(t)\phi_2(t)dt$$

Thus, $\int R(t)s_0(t)dt$ and $\int R(t)s_1(t)dt$ are the sufficient statistics.

Problem 4

(Minimum-Energy Signals)

(a) This part should be sufficiently self-explanatory. The average energy of a constellation with M signal points with probabilities $P_H(i), i \in [0, M-1]$ is given as $E = \mathbb{E}[\boldsymbol{S}^T\boldsymbol{S}] = \sum_{i=0}^{M-1} P_H(i)\|\boldsymbol{s}_i\|^2$, where $\boldsymbol{s}_i$ is the signal point vector corresponding to index i and $\boldsymbol{S}$ is the random vector corresponding to the transmitted signal point (when one dimensional, becomes a random variable).

(b) The average error probability doesn't depend on the value of $\boldsymbol{a}$ because the characterization of MAP or ML regions only depends upon the geometry of the constellation regardless of the translation in case the channel under consideration is an additive noise channel. This can be seen intuitively by the fact that, if the transmitted vector is $\boldsymbol{s}_i + \boldsymbol{a}$, the additive zero mean noise vector gets mean-shifted to $\boldsymbol{s}_i + \boldsymbol{a}$. The error event, given this translated point is transmitted, only depends upon the values that the noise vector can take i.e. the distance from the transmitted point to the hyperplanes dissecting the MAP or ML regions, which are also translated by the same amount $\boldsymbol{a}$. Hence while computing the distance between the translated point $\boldsymbol{s}_i + \boldsymbol{a}$ and the shifted hyperplane or neighbouring points, the components of $\boldsymbol{a}$ cancel correspondingly in the distance formula reducing the problem to the original case without any translation.

(c) From part (a), it is noticed that the average energy E changes with the amount of translation because it depends upon the location of each signal point in the constellation. Now, the required proof is as follows. If X is a random variable with zero mean, the variance of X, which roughly characterizes its energy, is written as $\mathbb{E}[X^2]$. Now, for a constant b,

$$\begin{aligned}\mathbb{E}[(X-b)^2] =& \mathbb{E}[X^2] + b^2 - 2b\mathbb{E}[X] \\ =& \mathbb{E}[X^2] + b^2 \\ \geq& \mathbb{E}[X^2] \end{aligned} \tag{1}$$

The last equation is valid because b^2 is a positive constant.

Let us consider our signal constellation vector to be $\boldsymbol{S}$ wherein the transmitter transmits one of the M points $\boldsymbol{S} = \boldsymbol{S}_i$ with probability p_i. The average energy is the expected value of the norm square of each constellation point. Now let $\boldsymbol{X} = \boldsymbol{S} - \mathbb{E}[\boldsymbol{S}]$ (in other words, $\boldsymbol{X}$ is the random vector corresponding to a translation of the original constellation wherein the translation takes the away any non-zero value of the centroid; not the geometric centroid, but the mean of the original constellation) and the average energy of this "centered" constellation is (vectors represented normally as column vector and their transposes are row vectors),

$$\begin{aligned}\mathbb{E}[\boldsymbol{X}^T\boldsymbol{X}] =& \mathbb{E}[(\boldsymbol{S} - \mathbb{E}[\boldsymbol{S}])^T(\boldsymbol{S} - \mathbb{E}[\boldsymbol{S}])] \\ =& \mathbb{E}[\boldsymbol{S}^T\boldsymbol{S}] - \mathbb{E}[\boldsymbol{S}]^T\mathbb{E}[\boldsymbol{S}] \\ \leq& \mathbb{E}[\boldsymbol{S}^T\boldsymbol{S}] \end{aligned} \tag{2}$$

Since we see from the above inequality that the random vector $\boldsymbol{X}$ has the least average energy among all translated constellations, the above constitutes a proof that a constellation with its centroid (mean) at the origin has the least energy of all possible translations. When the prior probabilities are equal, the mean is indeed the geometric centroid too.

Problem 5

(Antipodal Signaling)

(a) Assume for instance that $P_H(0) = P_H(1) = \frac{1}{2}$. Then, the decision regions are:

$$\begin{aligned}\mathcal{R}_0 &= \{(y_1, y_2) : y_2 < -y_1\}, \\ \mathcal{R}_1 &= \{(y_1, y_2) : y_2 \geq -y_1\}.\end{aligned}$$

If now, for instance, $Y_1 = a$, then for values of Y_2 that are larger than $-a$, we decide $\hat{H} = 1$, whereas for values of Y_2 that are smaller than $-a$, we decide $\hat{H} = 0$. Hence, we still need Y_2, and the knowledge of Y_1 is not sufficient.

(b) A new constellation for which Y_1 is a sufficient statistic is for instance

$$\begin{aligned}\tilde{\mathbf{s}}_0 &= (-a, 0), \\ \tilde{\mathbf{s}}_1 &= (a, 0).\end{aligned}$$

Problem 6

(Rectangular Waveforms)

(a) Plot out the curve of $s_1(t), \dots, s_4(t)$, we can show that:

$$\begin{aligned} s_1(t) &= 1 \cdot \phi_1(t) + 1 \cdot \phi_2(t) - 1 \cdot \phi_3(t) \\ s_2(t) &= -1 \cdot \phi_1(t) + 1 \cdot \phi_2(t) + 1 \cdot \phi_3(t) \\ s_3(t) &= 1 \cdot \phi_1(t) - 1 \cdot \phi_2(t) + 1 \cdot \phi_3(t) \\ s_4(t) &= -1 \cdot \phi_1(t) - 1 \cdot \phi_2(t) + 0 \cdot \phi_3(t) \end{aligned}$$

Thus the vector representations for them are:

$$\begin{aligned} s_1 &= [+1 \ +1 \ -1]^T \\ s_2 &= [-1 \ +1 \ +1]^T \\ s_3 &= [+1 \ -1 \ +1]^T \\ s_4 &= [-1 \ -1 \ 0]^T \end{aligned}$$

(b) We can compute the energy for each signal from its vector representation:

$$\begin{aligned} E_1 &= 3 \\ E_2 &= 3 \\ E_3 &= 3 \\ E_4 &= 2 \end{aligned}$$

Then, the average energy is

$$E_{\text{avg}} = \frac{3+3+3+2}{4} = \frac{11}{4}$$

(c) The result of distances d_{ij} between the signals is shown in Table 1.

	s_1	s_2	s_3	s_4
s_1	0	$2\sqrt{2}$	$2\sqrt{2}$	3
s_2	$2\sqrt{2}$	0	$2\sqrt{2}$	$\sqrt{5}$
s_3	$2\sqrt{2}$	$2\sqrt{2}$	0	$\sqrt{5}$
s_4	3	$\sqrt{5}$	$\sqrt{5}$	0

Table 1: Distances between the signals.

(d) The minimum distance $d_{\min} = \sqrt{5}$. The union bound is given by

$$P_e \leq 3Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = 3Q\left(\sqrt{\frac{5}{2N_0}}\right) = 3Q\left(\sqrt{\frac{10}{11} \cdot \frac{E_{\text{avg}}}{N_0}}\right) = 3Q\left(\sqrt{\frac{20}{11} \cdot \frac{E_b}{N_0}}\right)$$

For this part, you only need to have a correct expression inside the Q-function.

(e) The sketch is shown in Fig. 1.

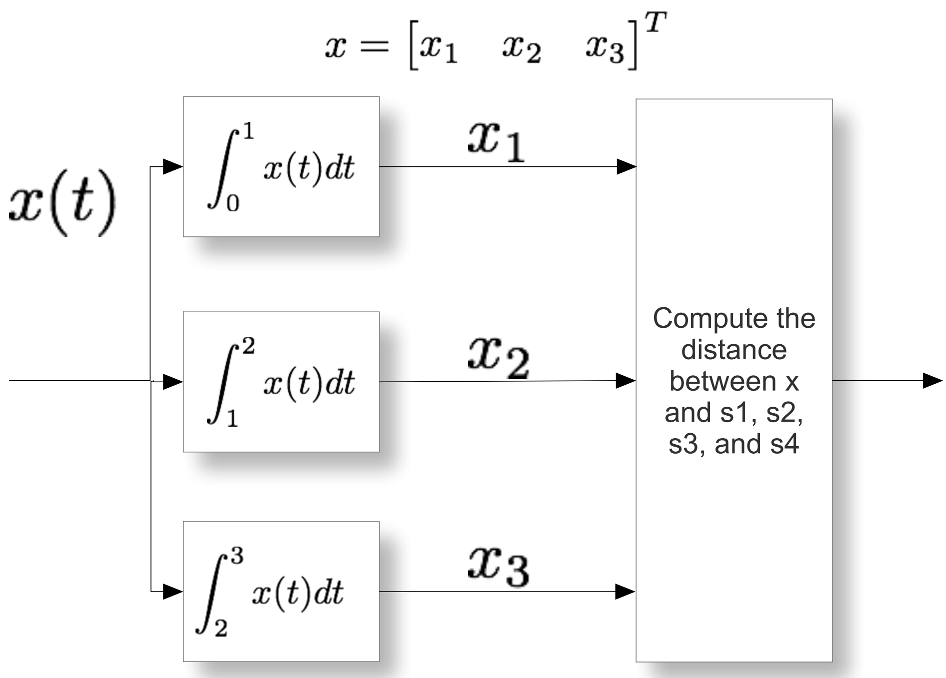


Figure 1: Sketch of the low complexity maximum likelihood receiver.

Problem 7

(Synchronization Importance)

(a) For the signal part we have $s_n = \int x(t)\psi(t - nT)dt = \mathcal{E}b_n$, where $\mathcal{E} = \int \psi(t)^2 dt$ is the energy of the pulse ψ. For the noise part we note that the noise at the output of the matched filter at time nT is $z_n = \int z(t)\psi(t - nT)dt$, where $z(t)$ is the additive noise of the channel. As $\{\psi(t - kT), \quad k \in \mathbb{Z}\}$ is an orthogonal collection, the z_n are i.i.d. Gaussian random variables with variance $\frac{N_0}{2}\mathcal{E}$.

(b) We have $r_n = b_n + z_n$. Hence, deciding on $b_n \in \{+1, -1\}$ is a binary hypothesis testing problem. Some further calculations show that the optimal decision rule is

$$r_n \underset{-1}{\overset{+1}{\gtrless}} 0,$$

and the error probability is $\mathbb{P}_e = Q(\sqrt{\frac{2\mathcal{E}}{N_0}})$.

(c) The output of the matched filter corresponding to the signal part is $y(t) = \sum b_i \phi(t - iT)$ where $\phi(t) = \psi(t) \star \psi(t)$ is a triangular pulse with peak $\mathcal{E}$ and width $2T$ centered around 0. At sampling time $t_n = nT - \tau$ we obtain

$$\begin{aligned} y_n &= \sum b_i \phi((n - i)T - \tau) \\ &= \mathcal{E}((1 - \frac{\tau}{T})b_n + \frac{\tau}{T}b_{n-1}). \end{aligned}$$

Hence, $\alpha = \mathcal{E}(1 - \frac{\tau}{T}), \quad \beta = \mathcal{E}\frac{\tau}{T}$.

(d) The decision rule compares the received value $r_n = y_n + z_n$ with 0 and decides that the value of b_n is $+1$ or -1. Assuming that $b_n = +1$ then $r_n = \mathcal{E}(1 - \frac{\tau}{T}) + \mathcal{E}\frac{\tau}{T}b_{n-1} + z_n$. By conditioning on b_n we have

$$\begin{aligned} Pr\{E\} &= \frac{1}{2}Pr\{r_n < 0|b_{n-1} = +1\} + \frac{1}{2}Pr\{r_n < 0|b_{n-1} = -1\} \\ &= \frac{1}{2}Pr\{\mathcal{E} + z_n < 0\} + \frac{1}{2}Pr\{\mathcal{E}(1 - \frac{2\tau}{T}) + z_n < 0\} \\ &= \frac{1}{2}\{Q(\sqrt{\frac{2\mathcal{E}}{N_0}}) + Q(\sqrt{\frac{2\mathcal{E}(1 - \frac{2\tau}{T})}{N_0}})\}. \end{aligned}$$

Problem 8

(More on matched filters...)

(a) The impulse response of the matched filter $s_{mf}(t) = s(T - t)$ is given by

$$s_{mf}(t) = \begin{cases} \frac{A}{T}(T - t)\cos(2\pi f_c(T - t)) & 0 \le t \le T, \\ 0 & \text{else.} \end{cases} \tag{3}$$

(b) The output of the matched filter, $y(t)\mid_{t=T}$, is the result of convolution (at time $t = T$) between the input signal $s(t)$ and the matched filter response $s_{mf}(t)$ which is given by the following set of equations:

$$\begin{aligned} y(t) &= s_{mf}(t) * s(t) && (4) \\ &= \int_0^\infty \left(\frac{A}{T}\right)^2 (T - \tau)\cos(2\pi f_c(T - \tau)) \cdot (t - \tau)\cos(2\pi f_c(t - \tau))\ \mathrm{d}\tau \\ \implies y(T) &= \left.\int_0^t \left(\frac{A}{T}\right)^2 (T - \tau)\cos(2\pi f_c(T - \tau)) \cdot (t - \tau)\cos(2\pi f_c(t - \tau))\ \mathrm{d}\tau\right|_{t=T} \\ &= \int_0^T \left(\frac{A}{T}\right)^2 (T - \tau)\cos(2\pi f_c(T - \tau)) \cdot (T - \tau)\cos(2\pi f_c(T - \tau))\ \mathrm{d}\tau. && (5) \end{aligned}$$

Simplifying and making change of variables as $T - \tau = t$, the above equation can be re-written as

$$y(T) = \int_0^T \left(\frac{A}{T}\right)^2 t^2\cos^2(2\pi f_c t)\ \mathrm{d}t. \tag{6}$$

(c) The correlator output is exactly the same as (6) which can be directly obtained from the definition of a correlator.

Problem 9

(Linear Block Coding)

(a) The parity check matrix is $cH^T = \mathbf{0}$, where $\mathbf{0}$ is a vector of size $n - k$ containing all zeros.

(b) This can be solved by enumerating all the combinations of missing values in column 2 and column 3 that satisfy $cH^T = 0$. By analyzing the nonzero elements of the received codeword, it is only the modulo-2 sum of rows two, three, six, and seven of H^T that contribute to producing the all-zero syndrome ensuring c is a valid codeword. Since there are four missing elements, there are sixteen possible ways to fill these elements with ones and zeros. However, only four of these sixteen cases will produce $cH^T = 0$ given the valid codeword in the problem statement. These four are:

$$\begin{aligned}[A, B, C, D] &= [1, 0, 1, 0] \\ &= [0, 1, 0, 1] \\ &= [0, 1, 1, 0] \\ &= [1, 0, 0, 1]\end{aligned}$$

(c) Starting with the four possibilities found in part (b), we want to find which of these has the maximum d_{min}. Since the distances separating codewords do not depend on which codeword is used as the reference, for convenience it is easiest to use the all-zero codeword as a reference. For a given code, d_{min} will be the number of ones in the codeword which has the fewest number of ones. From H, construct the generator matrix, G, as:

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & A & C \\ 0 & 0 & 1 & 1 & 1 & B & D \end{bmatrix}$$

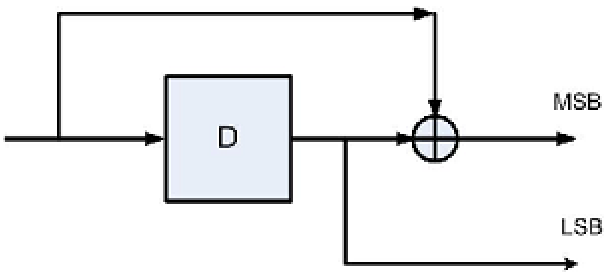
From this we can see that there are two sets of values that ensure that all rows of G have a Hamming weight of at least 4:

$$\begin{aligned}[A, B, C, D] &= [0, 1, 1, 0] \\ &= [1, 0, 0, 1]\end{aligned}$$

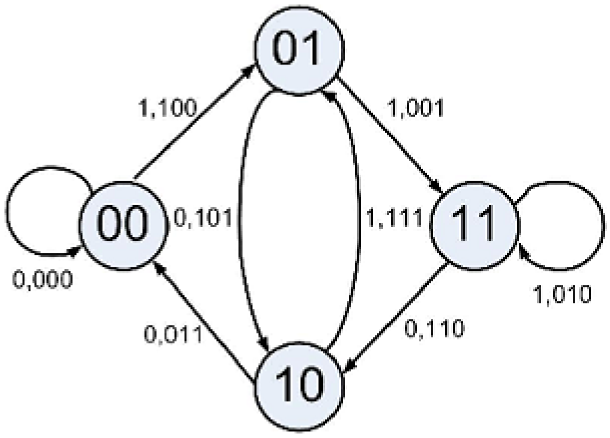
Problem 10

(Convolutional Coding)

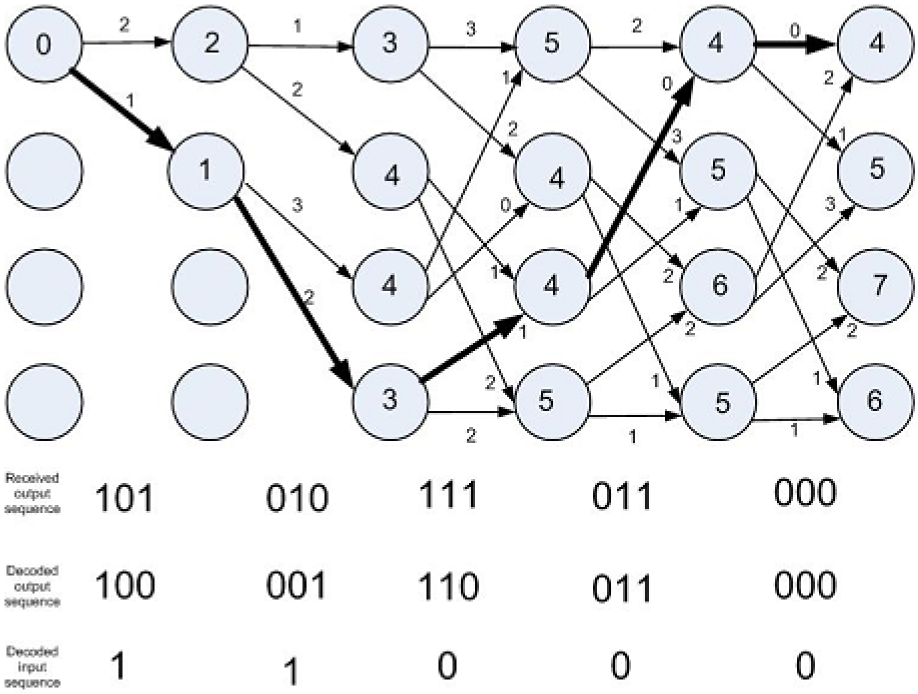
(a) The block diagram is seen below:



(b) The state diagram for the encoder with rate 1/3 is shown below:



According to the trellis diagram shown below, the decoded input sequence is [1 1 0 0 0].



Problem 11

(Convolutional Code)

(a) An implementation of the encoder is shown in Fig. 2:

(b) The rate of this code is 2/3: Two inputs produce three outputs.

(c) The state diagram is very similar to the one considered in class. The difference is that now, *every* state is accessible from every other state. We use the following terminology: the state label is a, b, where a is the "state of the even sub-sequence", i.e. contains d_{2n-2}, and b is the "state of the odd sub-sequence", i.e. contains d_{2n-1}. On the arrows, we only mark the outputs; the input required to make a particular transition is simply the next state, therefore we omitted it. Take for example the arrow linking state $1, 1$ to state $-1, 1$.

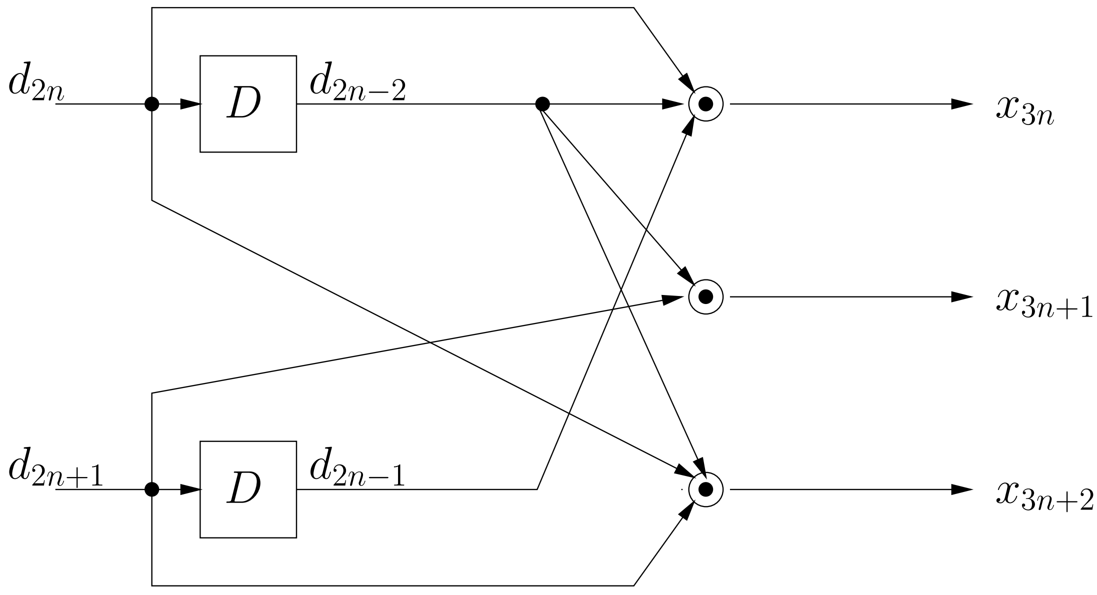
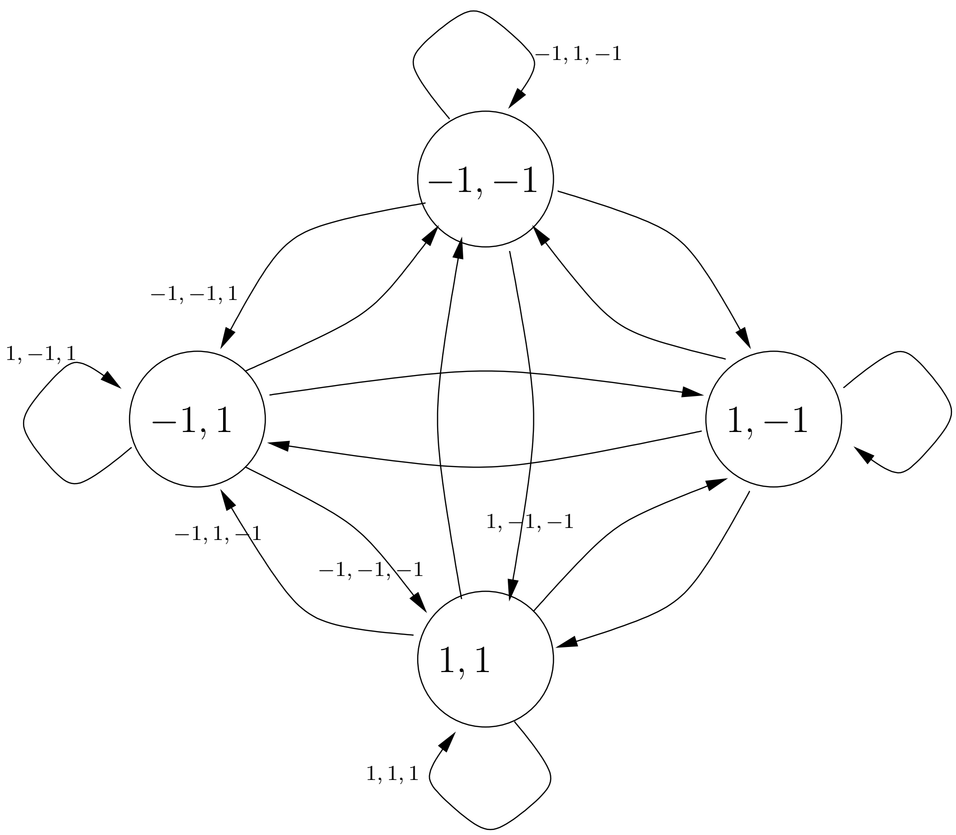


Figure 2: Implementation of the encoder

It is clear that the inputs have to be $d_{2n} = -1$ and $d_{2n+1} = 1$. The corresponding output in order $x_{3n}, x_{3n+1}, x_{3n+2}$ are marked on the branch: $-1, 1 - 1$. We only give a few of the labels in the figure below; it should be easy to find the rest.

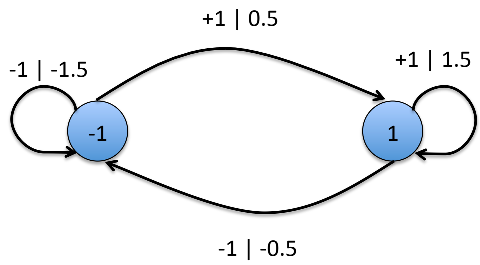


Problem 12

("Real" convolutional decoding)

(a) The possible values of y_t are $-\frac{3}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$.

(b) We plot the state diagram below. Then, we can conclude that the input sequence is $(+1,\ +1,\ +1)$.



(c) The 8 possible output sequences are:

(s_0, s_1, s_2)	(y_0, y_1, y_2, y_3)
$(-1, -1, -1)$	$(-1.5, -1.5, -1.5, -1.5)$
$(-1, -1, +1)$	$(-1.5, -1.5, +0.5, -0.5)$
$(-1, +1, -1)$	$(-1.5, +0.5, -0.5, -1.5)$
$(-1, +1, +1)$	$(-1.5, +0.5, +1.5, -0.5)$
$(+1, -1, -1)$	$(+0.5, -0.5, -1.5, -1.5)$
$(+1, -1, +1)$	$(+0.5, -0.5, +0.5, -0.5)$
$(+1, +1, -1)$	$(+0.5, +1.5, -0.5, -1.5)$
$(+1, +1, +1)$	$(+0.5, +1.5, +1.5, -0.5)$

(d) The trellis has 2 states and 4 stages. The state means the value of s_{t-1}, i.e., the previous transmitted symbol that is memorized by the channel. The paths corresponding to different sequences are shown in Figure 3.

(e) Due to conditional independence of $\{y_t\}$ given $\{s_t\}$, the conditional pdf is given by

$$f_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{s}) = \frac{1}{(2\pi\sigma^2)^2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{t=}^{3} |y_t - s_t - \frac{1}{2}s_{t-1}|^2 \right\}$$

Taking natural logorithm, we obtain the log-likelihood function in the problem.

(f) Since all sequences are equally likely, the MAP rule becomes $\arg\max_{\boldsymbol{s}} \log f_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{s})$, which leads to the rule given in the problem statement.

(g) The branch metric for each edge should be computed from $|y_t - s_t - \frac{1}{2}s_{t-1}|^2$ and we need to find the path that has the minimum aggregate branch metric. The optimal path in the trellis is shown in the following figure 4. The decoded message should be $(s_{-1},\ s_0,\ s_1,\ s_2,\ s_3) = (-1, +1, -1, +1, -1)$.

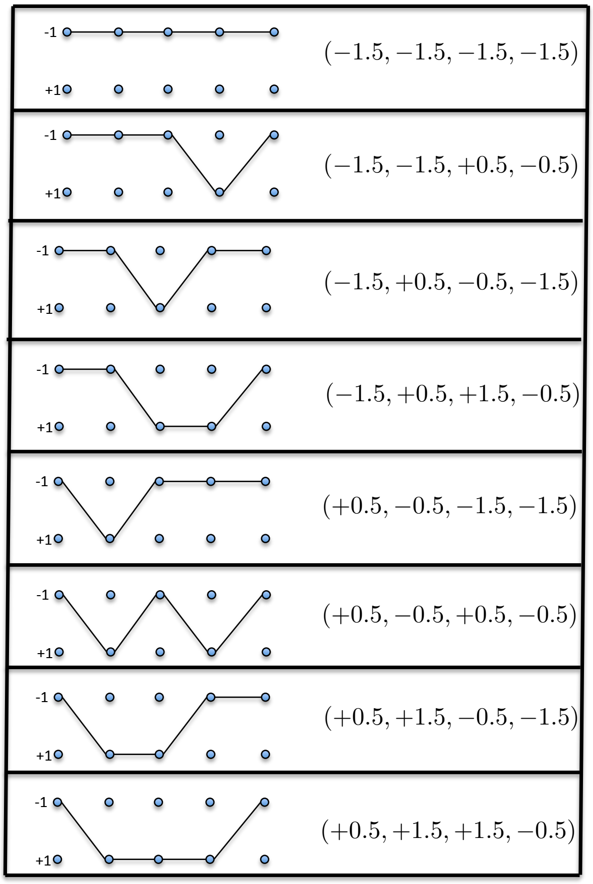


Figure 3: Trellis and paths for sequences.

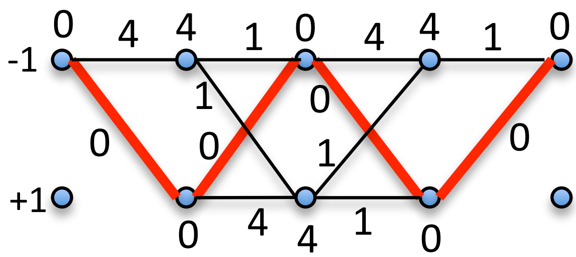
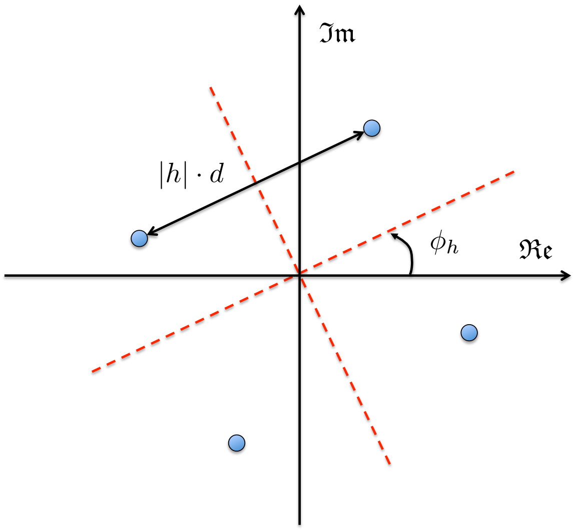


Figure 4: Optimal paths for the decoded sequence.

Problem 13

(Fading channels)

(a) When h is a complex number, it can be written as $h = |h|e^{j\phi_h}$, where ϕ_h denotes the phase of h. Therefore, the effect of $h \cdot s[k]$ is rotating the constellation by ϕ_h and scaling it by $|h|$. If the receiver knows h perfectly, then we know the rotated constellation shown in the following figure with the optimal decision boundaries.



(b) The error probability is given by

$$
\begin{aligned}
P_e &= 1 - \left[1 - Q\left(\frac{\frac{|h| \cdot d}{2}}{\sqrt{\frac{\sigma^2}{2}}}\right)\right]^2 \\
&= 1 - \left[1 - Q\left(\sqrt{\frac{|h|^2 \cdot d^2}{2\sigma^2}}\right)\right]^2 \\
&= 2Q\left(\sqrt{\frac{|h|^2 \cdot d^2}{2\sigma^2}}\right) - \left[Q\left(\sqrt{\frac{|h|^2 \cdot d^2}{2\sigma^2}}\right)\right]^2 \\
&\approx 2Q\left(\sqrt{\frac{|h|^2 \cdot d^2}{2\sigma^2}}\right)
\end{aligned}
$$

(c) Denote $\xi_0 = \frac{d^2}{4\sigma^2}$. Then, the probability of error given $|h|$ is

$$P_e(|h|^2) \approx 2Q(\sqrt{|h|^2 \xi_0})$$

and the average probability of error is

$$
\begin{aligned}
\bar{P}_e &\approx \mathbb{E}_{|h|^2}\left\{P_e(|h|^2)\right\} \\
&= \mathbb{E}_{|h|^2}\left\{2Q(\sqrt{|h|^2 \xi_0})\right\} \\
&= 2\mathbb{E}_{|h|^2}\left\{Q(\sqrt{|h|^2 \xi_0})\right\} \\
&= 1 - \sqrt{\frac{\xi_0}{\xi_0 + 1}} \\
&= 1 - \sqrt{\frac{1}{\xi_0^{-1} + 1}}
\end{aligned}
$$

(d)

$$\bar{P}_e \approx 1 - \sqrt{\frac{1}{\xi_0^{-1} + 1}}$$

$$\approx 1 - \left(1 - \frac{1}{2}\frac{1}{\xi_0}\right)$$
$$= \frac{1}{2\xi_0} = 2 \cdot \frac{1}{d^2/\sigma^2}$$

For the transmitted constellation, the signal energy E is given by

$$E = \frac{d^2}{2}$$

Therefore, the error probability can be expressed as

$$\bar{P}_e = \frac{1}{\frac{E}{\sigma^2}}$$

To achieve error probability of 10^{-3}, we need to have

$$\frac{1}{\frac{E}{\sigma^2}} = 10^{-3} \quad \Rightarrow \quad \frac{E}{\sigma^2} = 10^3 = 30\text{dB}$$

Problem 14

(*OFDM System*)

(a) The impulse response of the channel has a length of two. Therefore, to ensure ISI-free transmission, the length of the cyclic prefix should be at least one.

(b) Since $N = 4$, we need to divide the data packet into two blocks:

$$1, -1, 1, -1$$
$$1, 1, 1, 1$$

Taking N-point IDTFT for each block, we obtain:

$$0, 0, 1, 0$$
$$1, 0, 0, 0$$

where the IDTFT formula is:

$$X[k] = \frac{1}{4}\sum_{n=0}^{3} x[n] e^{j\frac{2\pi}{4}kn}$$

Then, the entire output sequence after adding cyclic prefix would be

$$\underline{0}, 0, 0, 1, 0, \underline{0}, 1, 0, 0, 0$$

where the digits with underlines are the cyclic prefixes.

(c) We can evaluate $\tilde{H}[m]$ as

$$\tilde{H}[m] = H(z)\big|_{z=e^{j\frac{2\pi m}{N}}}$$
$$= 1 + e^{-j\frac{2\pi m}{N}}$$
$$= 1 + e^{-j\frac{2\pi m}{4}}$$

i.e.,

$$\begin{aligned}\tilde{H}[0] &= 2\\ \tilde{H}[1] &= 1 - j\\ \tilde{H}[2] &= 0\\ \tilde{H}[3] &= 1 + j\end{aligned}$$

Problem 15

(Cyclic prefix)

(a) The channel has two taps $h_0 = 1$ and $h_1 = 0.9$. After convolving $s[k]$ with $h[k]$, we get the channel model:

$$y[k] = h_0 s[k] + h_1 s[k-1] + z[k] = s[k] + 0.9s[k-1] + z[k]$$

If we include the ISI term, $0.9s[k-1]$, with the noise, thus ignoring ISI, the effective noise power is the power from $z[k]$ plus the power from the ISI. We will call this noise power σ^2, which is equal to:

$$\sigma^2 = 0.9^2 E + N_0/2$$

where E is the average energy of the signal (which is invariant to time-shifts) and $N_0/2$ is the power from $z[k]$. Thus, the probability of error is given by

$$P_e = Q(\frac{d}{2\sigma}) = Q(\frac{1}{2}\sqrt{\frac{E}{0.81E + N_0/2}})$$

(b) Taking the expected value of the energy of each term, as we did in class, we can derive an expression for the increase in energy caused by this "naive" precoding method. For the system with two taps in this problem, the derivation simplifies to

$$\mathbb{E}[|S_n|^2] = \mathcal{E}(1 + \frac{|h_1|^2/|h_0|^2}{1 - |h_1|^2/|h_0|^2})$$

After plugging in values $h_0 = 1$ and $h_1 = 0.9$,

$$\begin{aligned}\mathbb{E}[|S_n|^2] &= \mathcal{E}(1 + \frac{|0.9|^2/|1|^2}{1 - |0.9|^2/|1|^2})\\ &= 5.26\mathcal{E}\end{aligned}$$

So the transmitter must send 5.26 times as much energy as without precoding.

(c) The length of the cyclic prefix is equal to the "memory" of the channel. This channel has memory from one previous symbol, so the prefix is of length 1.

(d) For $N_c = 2$, the cyclic prefix is of length 1, so we send 3 bits for every 2 data bits, so the reduction in rate is 2/3.
Since we now send 3 bits instead of 2, the excess power is 3/2.

For $N_c = 4$, the cyclic prefix is of length 1, so we send 5 bits for every 4 data bits, so the reduction in rate is 4/5.
Since we now send 5 bits instead of 4, the excess power is 5/4.

(e) Since the error rate is calculated for the same energy per information bit, the energy in the case of OFDM will be $E_{OFDM} = (\frac{N_c}{N_c+1})E$. The energy factors into the distance between signals in the error probability. The transfer function of each subchannel also attenuates the signal, so that

$$E_i = |H_i|^2 \cdot E$$

The ith channel has the form

$$H_i = \sum_{l=0}^{L-1} h[l]e^{-\frac{j2\pi il}{N_c}} = 1 + 0.9e^{-\frac{j2\pi i}{N_c}}$$

For $N_c = 2$, the channels are $H_0 = 1 + 0.9 = 1.9$ and $H_1 = 1 - 0.9 = 0.1$, so

$$\begin{aligned} P_e &= \frac{1}{2}Q\left(\frac{1}{2}\sqrt{\frac{1.9^2(\frac{N_c}{N_c+1})E}{\frac{N_0}{2}}}\right) + \frac{1}{2}Q\left(\frac{1}{2}\sqrt{\frac{0.1^2(\frac{N_c}{N_c+1})E}{\frac{N_0}{2}}}\right) \\ &= \frac{1}{2}Q\left(\sqrt{\frac{3.61E}{3N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{0.01E}{3N_0}}\right) \end{aligned}$$

The factor of $\frac{1}{2}$ inside the Q function is from the factor $\frac{d}{2}$ in the general form $Q(\frac{d}{2\sigma})$. For $N_c = 4$, the channels are

$$\begin{aligned} H_0 &= 1.9 \\ H_1 &= 1 + 0.9e^{-\frac{j\pi}{2}} \\ H_2 &= 0.1 \\ H_3 &= 1 + 0.9e^{-\frac{j3\pi}{2}} \end{aligned}$$

The magnitude-squared values, $|H_i|^2$ are then

$$\begin{aligned} |H_0|^2 &= 1.9^2 = 3.61 \\ |H_1|^2 &= 1^2 + 0.9^2 = 1.81 \\ |H_2|^2 &= 0.1^2 = 0.01 \\ |H_3|^2 &= 1^2 + 0.9^2 = 1.81 \end{aligned}$$

Then,

$$\begin{aligned} P_e &= \sum_{i=0}^{3} \frac{1}{4}Q\left(\frac{1}{2}\sqrt{\frac{|H_i|^2(\frac{N_c}{N_c+1})E}{\frac{N_0}{2}}}\right) \\ &= \frac{1}{4}Q\left(\sqrt{3.61\frac{2E}{5N_0}}\right) + \frac{1}{2}Q\left(\sqrt{1.81\frac{2E}{5N_0}}\right) + \frac{1}{4}Q\left(\sqrt{0.01\frac{2E}{5N_0}}\right) \\ &= \frac{1}{4}Q\left(\sqrt{\frac{7.22E}{5N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{3.62E}{5N_0}}\right) + \frac{1}{4}Q\left(\sqrt{\frac{0.02E}{5N_0}}\right) \end{aligned}$$