Practice Problem Set # 2

Problem 1

(Autocorrelation and PSD)

Consider a zero-mean wide-sense stationary process $X(t)$ with autocorrelation function $R_X(\tau) = \mathbb{E}[X(t)X(t+\tau)] = e^{-2|\tau|}$. Let $N(t)$ be another zero-mean wide-sense stationary process that is independent of $X(t)$ with autocorrelation $R_N(\tau) = \mathbb{E}[N(t)N(t+\tau)] = e^{-3|\tau|}$.

(a) Find the power spectral density $S_X(f)$ of process $X(t)$.

(b) Find the power spectral density $S_N(f)$ of process $N(t)$.

(c) Let $Y(t) = X(t) + N(t)$. Find the autocorrelation function of $Y(t)$.

(d) What is the power spectral density $S_Y(f)$ of $Y(t)$?

Problem 2

(Short questions)

For the following questions, give a short explanation/justification for your answers.

(a) Assume that in a binary hypothesis testing problem we design the optimal decision rule assuming that $p_0 = p_1 = \frac{1}{2}$ and we find that for this decision rule the conditional error probabilities are equal, i.e., $P(E|H_0) = P(E|H_1)$. Assume now that p_0 and p_1 change, and we do *not* change the decision rule. How does the (total) error probability change?

(b) Consider an orthogonal signaling scheme over an AWGN channel which transmits one bit. The two corresponding waveforms are $\psi_0(t)$ and $\psi_1(t)$, where ψ_0 and ψ_1 are orthogonal equi-energy waveforms. Does the shape of the waveforms have an effect on the error probability?
[YES / NO] Please justify your answer.

(c) Consider transmission over an AWGN channel using a convolutional code. Let Y be the output of the channel observed at the receiver (this is a whole sequence). Consider the optimal "path" $\tilde{X}$ chosen by the Viterbi decoder given Y, i.e., the sequence of bits with the largest metric given Y. Let X_i be the bit sent during the i-th step of the transmission. Does this imply that $P(X_i = \tilde{X}_i \mid Y)$ is maximized?
[YES / NO] Please justify your answer.

(d) Consider transmission over an equivalent discrete time channel

$$y[k] = s[k] * h[k] + z[k] = \sum_{n=0}^{L} h[n]s[k-n] + z[k]$$

where $z[k]$ is AWGN noise. Suppose that the designer of an OFDM signal thinks that the equivalent discrete time channel is described by the above model with $L = 2$, and

$$H(z) = 1 + 0.9z^{-1} + 0.8z^{-2}.$$

Using this model, the designer chooses a cyclic prefix of length 2. However, it turns out that the actual channel has $L = 1$ and is

$$H(z) = 1 + 0.6z^{-1}$$

Does the OFDM transmit and receive design fail because of this modeling error?
[YES / NO] Please justify your answer.

Problem 3

(Short questions)

For the following questions, give a short explanation/justification for your answers.

(a) For orthogonal signalling with equiprobable symbols $p_H(0) = p_H(1) = \frac{1}{2}$ which results in constellations $\mathbf{s}_0 = \begin{bmatrix} \sqrt{E} \\ 0 \end{bmatrix}$ and $\mathbf{s}_1 = \begin{bmatrix} 0 \\ \sqrt{E} \end{bmatrix}$. This constellation has the lowest power for the same error probability and rate. [YES / NO]

(b) Consider binary messages ($M = 2$) transmitted over a AWGN waveform channel, *i.e.,* $R(t) = x(t) + N(t)$, where $x(t) = s_i(t)$ if $H = i \in \{0, 1\}$, where $s_0(t), s_1(t)$ are *not* orthogonal. Then $\int R(t)s_0(t)dt$ and $\int R(t)s_1(t)dt$ form a set of sufficient statistics. [YES / NO]

Problem 4

(Minimum-Energy Signals)

Consider a given signal constellation consisting of vectors $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_m\}$. Let signal $\boldsymbol{s}_i$ occur with probability p_i. In this problem, we study the influence of moving the origin of the coordinate system of the signal constellation under an *additive noise channel.* That is, we study the properties of the signal constellation $\{\boldsymbol{s}_1 - \boldsymbol{a}, \boldsymbol{s}_2 - \boldsymbol{a}, \ldots, \boldsymbol{s}_m - \boldsymbol{a}\}$ as a function of $\boldsymbol{a}$.

(a) Draw a sample signal constellation, and draw its shift by a sample vector $\boldsymbol{a}$. Also find the average energy of the constellation in both cases.

(b) Does the average error probability, P_e, depend on the value of $\boldsymbol{a}$? Explain.

(c) The average energy per symbol depends on the value of $\boldsymbol{a}$. For a given signal constellation $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_m\}$ and given signal probabilities p_i, prove that the value of $\boldsymbol{a}$ that minimizes the average energy per symbol is the centroid (the center of gravity) of the signal constellation, i.e.,

$$\boldsymbol{a} \;=\; \sum_{i=1}^{m} p_i \boldsymbol{s}_i. \tag{1}$$

Hint: First prove that if X is a real-valued zero-mean random variable and $b \in \mathbb{R}$, then $E[X^2] \leq E[(X - b)^2]$ with equality iff $b = 0$. Then extend your proof to vectors and consider $\mathbf{X} = \boldsymbol{S} - E[\boldsymbol{S}]$ where $\mathbf{S} = \mathbf{S}_i$ with probability p_i.

Problem 5

(Antipodal Signaling)

Consider the signal constellation consisting of the two points $\mathbf{s}_0 = (a, a)$ and $\mathbf{s}_1 = (-a, -a)$. Assume that $\mathbf{s}_0$ and $\mathbf{s}_0$ are used for communication of a Gaussian vector channel. More precisely:

$$H = 0 : \mathbf{Y} = \mathbf{s}_0 + \mathbf{Z},$$
$$H = 0 : \mathbf{Y} = \mathbf{s}_1 + \mathbf{Z},$$

where $Z\ \mathbb{N}(\mathbf{0}, \sigma^2 I_2)$. Hence, $\mathbf{Y}$ is a vector with two components $\mathbf{Y} = (Y_1, Y_2)$.

(a) Argue that Y_1 is *not* a sufficient statistic.

(b) Give a different signal constellation with two signals $\hat{\mathbf{s}}_0$ and $\hat{\mathbf{s}}_1$ such that, when used in the above communication setting, Y_1 is a sufficient statistic.

Problem 6

(Rectangular Waveforms)

Consider the following set of digital waveforms:

$$s_1(t) = \{1 \text{ over } (0, 2), -1 \text{ over } (2, 3), 0 \text{ otherwise } \}$$
$$s_2(t) = \{ -1 \text{ over } (0, 1), 1 \text{ over } (1, 3), 0 \text{ otherwise } \}$$
$$s_3(t) = \{1 \text{ over } (0, 1) \text{ and } (2, 3), -1 \text{ over } (1, 2), 0 \text{ otherwise } \}$$
$$s_4(t) = \{ -1 \text{ over } (0, 2), 0 \text{ otherwise}\}$$

(a) Draw the waveforms of $s_1(t), \ldots, s_4(t)$. Show that the function may be expressed as linear combinations of the following basis functions, and express each signal as a vector:

$$\phi_1(t) = \{1 \text{ over } (0, 1), 0 \text{ otherwise}\}$$
$$\phi_2(t) = \{1 \text{ over } (1, 2), 0 \text{ otherwise}\}$$
$$\phi_3(t) = \{1 \text{ over } (2, 3), 0 \text{ otherwise}\}$$

(b) Compute the average energy E_{av} of the signal set.

(c) Compute the set of distances $\{d_{ij}\}$ between signals $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$, $i = 1, ..., 4,\ j = 1, ..., 4.$

(d) Give nearest neighbor union bound (NNUB) for the probability of error as a function of E_b/N_0, assuming an additive white Gaussian noise (AWGN) channel with power spectral density (PSD) $\frac{N_0}{2}$. E_b denotes average energy per bit: $E_b = E_{\text{av}}/\log_2(M)$, where M is the number of points in the constellation plot.

(e) Sketch a low complexity maximum likelihood receiver for this signal set.

Problem 7

(Synchronization Importance)

In your project, you will notice that synchronization is an important factor for successful communication. Even the best synchronization algorithms are not fully accurate and have some residual timing error in the presence of noise. In this problem, we are going to investigate the effect of this residual timing error on the error probability.

Assume that we send a sequence of equi-probable $\{+1, -1\}$ valued bits $b_0, b_1, \cdots$ from the transmitter to the receiver over an AWGN channel with power spectral density $\frac{N_0}{2}$. For simplicity, suppose that we are using a rectangular pulse for transmission. In other words, the transmitted signal is $x(t) = \sum_{i=0}^{\infty} b_i \psi(t - iT)$, where T is the transmission period and

$$\psi(t) = \begin{cases} \sqrt{\frac{\mathcal{E}}{T}}, & \text{if } 0 \le t \le T, \\ 0, & \text{otherwise.} \end{cases}$$

To extract the bits, we use the matched filter $h(t) = \psi(-t)$ and, assuming that we have perfect synchronization, we sample the output of the matched filter at times $nT, \quad n = 0, 1, \cdots$.

(a) Write an expression for the output of the matched filter at time nT, denoted by r_n. Your expression should be in terms of the signal part and the noise part. Also, show that the signal part only depends on the bit transmitted at time nT, i.e. b_n, and the noise part is a sequence of i.i.d. Gaussian random variables.

(b) Derive the optimal decision rule for identifying b_n using r_n and find its error probability as a function of $\frac{\mathcal{E}}{N_0}$.

(c) Now assume that there is a timing error in sampling the output of the matched filter. In other words, instead of sampling at time nT we sample at $nT - \tau$, where $0 \le \tau \le T$ is called the sampling jitter. Obtain an expression for the signal part of r_n in terms of the transmitted bit sequence. **Hint:** Your expression should be of the form $\alpha b_n + \beta b_{n-1}$, where α and β depend on $\frac{\tau}{T}$. A sketch of the output of the matched filter may be helpful.

(d) Now assume that we have the previous decision rule and it works without any knowledge of the timing error. Try to obtain the probability of error in deciding b_n, $n \ge 1$, from r_n. **Hint:** This error probability now depends also on b_{n-1}. Use conditioning on b_{n-1} to simplify the calculations. Also, notice that because of symmetry you can assume that $b_n = 1$ is transmitted.

Problem 8

(More on matched filters...)

Consider the signal

$$s(t) = \begin{cases} \frac{A}{T} t \cos 2\pi f_c t & 0 \le t \le T, \\ 0 & \text{else.} \end{cases} \tag{2}$$

(a) Determine the impulse response of the matched filter for the signal.

(b) Determine the output of the matched filter at $t = T$ when $s(t)$ is passed through it.

(c) Suppose the signal $s(t)$ is passed through a correlator that correlates the input $s(t)$ with $s(t)$, determine the value of the correlator output at $t = T$. Compare your result with that in (b).

Problem 9

(Linear Block Coding)

(a) In a linear block code, if c is known to be a valid codeword, what is the result of cH^T, where H is the parity check matrix? Your answer should specify whether the result of multiplying cH^T is a scalar, vector or matrix. If it is a vector or matrix, specify its dimensions.

(b) Consider a systematic (7,3) code with the following parity check matrix. Note that there are four unknown binary elements A, B, C, D:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & A & B & 0 & 0 & 1 & 0 \\ 1 & C & D & 0 & 0 & 0 & 1 \end{bmatrix}$$

Suppose that $c = [0110011]$ is known to be a valid codeword. Proved all the possible valid solutions for $[A, B, C, D]$.

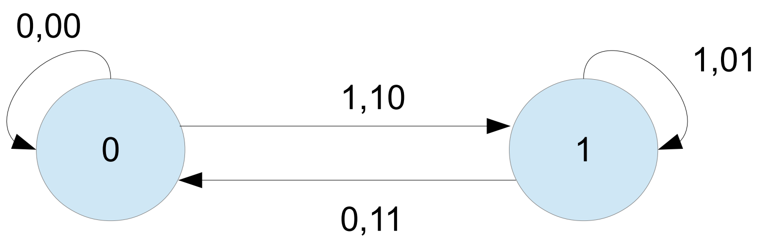
(c) Provide all the possible valid solutions for $[A, B, C, D]$ that satisfy both of the following conditions:

1. The code includes c as one of the codewords.
2. There is no solution for $[A, B, C, D]$ that gives a higher d_{min}, where d_{min} is the smallest Hamming distance from any codeword to any other codeword in the code.

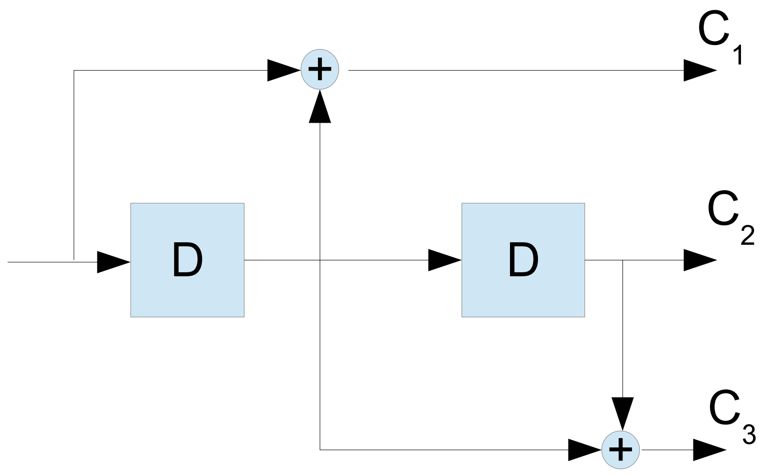
Problem 10

(Convolutional Coding)

(a) Design a convolutional encoder that corresponds to the state diagram shown below. Each branch in the state diagram below is labeled with the input bit (to the left of the comma) and the corresponding output bits (to the right of the comma). Of the two output bits, the left bit is the MSB (most-significant bit) and the right bit is the LSB. Your answer must be in the form of a block diagram of the encoder you design. The block diagram should clearly show any delay elements, adders, and should include clear labels indicating the MSB and LSB.



(b) Now consider the convolutional encoder shown in block diagram form below. Every time a new input bit is provided to this encoder, a triple of output bits in the form $C_1C_2C_3$ is produced. Suppose that this encoder is used to encode a sequence which is transmitted through a channel and received as $[101, 010, 111, 011, 000]$. Assume that the encoder started with both delay elements containing zero. Construct a trellis and perform the Viterbi algorithm to determine the corresponding input sequence. Your answer should clearly show the trellis, calculation of path metrics, identification of the minimum weight path, and of course the input sequence. Note that "input sequence" in this problem refers to the sequence of bits provided at the input to the convolutional coder, before encoding, and before possible corruption during transmission over the channel.



Problem 11

(Convolutional code)
The following equations define a convolutional code for a data sequence $d_i \in \{-1, 1\}$:

$$x_{3n} = d_{2n} \cdot d_{2n-1} \cdot d_{2n-2} \quad (3)$$
$$x_{3n+1} = d_{2n+1} \cdot d_{2n-2} \quad (4)$$
$$x_{3n+2} = d_{2n+1} \cdot d_{2n} \cdot d_{2n-2} \quad (5)$$

(a) Draw an implementation of the encoder of this convolutional code, using only delay elements D and multipliers. *Hint:* Split the data sequence d into two sequences, one containing only the even-indexed samples, the other containing only the odd-indexed samples.

(b) What is the rate of this convolutional code?

(c) Draw the state diagram for this convolutional encoder.

Problem 12

("Real" convolutional decoding)
Consider a discrete channel which has a binary input sequence, *i.e.,* a sequence of elements in $\{-1, +1\}$, of length 3, call the symbols s_0, s_1, s_2. At the receiver we observe the real-valued sequence $y_0, \cdots, y_3$ of length 4. The channel is of the form

$$y_t = s_t + \frac{1}{2} s_{t-1} + Z_t,$$

for $t = 0, 1, 2, 3$, where Z_t is a sequence of i.i.d. Gaussian noise samples of mean zero and variance σ^2, and where $s_{-1} = s_3 = -1$. In words, the channel has *memory, i.e.,* the output at time t depends on the input at time t as well as on the input at time $t - 1$.

(a) Assume at first that $\sigma^2 = 0$, *i.e.,* there is no noise. What are the possible values of the observation, *i.e.,* the possible values for y_t?

(b) Assume that $\sigma^2 = 0$ and that you observe at the output the sequence $(0.5,\ 1.5,\ 1.5,\ -0.5)$. What is the corresponding input sequence (s_0, s_1, s_2)?

(c) If we assume $\sigma^2 = 0$, write down the 8 possible output sequences (y_0, y_1, y_2, y_3)?

(d) Write down these 8 sequences as paths in a trellis. How many states does the trellis have and what is the meaning of the state?

(e) Assume now that we have a general σ^2 and that the symbols have equal probabilities (hence also the 8 possible output sequences are equally likely). Given (y_0, y_1, y_2, y_3), we want to find the most likely input sequence (s_0, s_1, s_2). We will derive optimal decision criterion for this problem.

Since

$$y_t = s_t + \frac{1}{2}s_{t-1} + Z_t,$$

where $\{Z_t\}$ is i.i.d. $\eta(0, \sigma^2)$ and denote $\mathbf{y} = (y_0, y_1, y_2, y_3)$ and $\mathbf{s} = (s_0, s_1, s_2, s_3)$. Show that

$$\log f_{\mathbf{Y}|\mathbf{S}}(\mathbf{y}|\mathbf{s}) = -\frac{1}{2\sigma^2}\sum_{t=0}^{3}|y_t - s_t - \frac{1}{2}s_{t-1}|^2 - 2\log(2\pi\sigma^2).$$

(f) Show that the optimal decision rule is,

$$\arg\min_{\mathbf{s}} \sum_{t=0}^{3}|y_t - s_t - \frac{1}{2}s_{t-1}|^2. \tag{6}$$

(g) Suppose the received sequence $(y_0, y_1, y_2, y_3) = (0.5,\ -0.5,\ 0.5,\ -0.5)$. Find the optimal transmitted sequence (efficiently). Demonstrate the efficient method you used to solve this problem.
Hint: How can you use (6) in a Viterbi decoding algorithm for the real convolution channel?

Problem 13

(*Fading Channels*)
Suppose we want to send a QPSK signal constellation, illustrated in Figure 1 over a slowly fading wireless channel. Specifically, we use $s[k] \in \{\pm\frac{1}{2}d \pm j\frac{1}{2}d\}$ to represent different messages $\{0, 1, 2, 3\}$. The equivalent discrete-time channel as seen in the lectures, can be modeled as:

$$y[k] = hs[k] + z[k], \tag{7}$$

where $z[k] = z^I[k] + jz^Q[k]$ is complex additive Gaussian noise, with $z^I[k] \sim \eta(0, \frac{\sigma^2}{2})$, $z^Q[k] \sim \eta(0, \frac{\sigma^2}{2})$, $z^I[k]$ and $z^Q[k]$ being independent of each other.

(a) Suppose the channel h is a *fixed* and known constant. Then find the optimal decision regions to minimize error probability.

(b) Find the error probability for the decision rule in (a), when the channel h is a fixed constant and known.

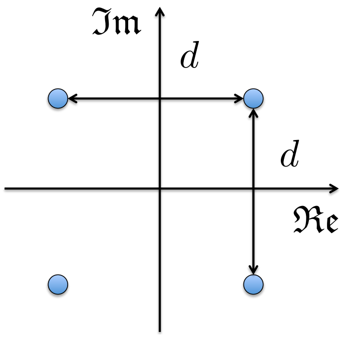


Figure 1: The QPSK constellation.

(c) Now suppose we have a statistical model for the wireless channel. The fading channel h is statistically modeled as a complex Gaussian channel, and therefore $|h|^2$ is exponentially distributed, *i.e.,*

$$f_{|h|^2}(a) = e^{-a}, \quad a \geq 0. \tag{8}$$

Since the channel is assumed to be known to the receiver, the optimal decision rule remains the same as in (a). Find the average error probability, *i.e.,* $\mathbb{E}_{|h|^2}[P_e(|h|^2)]$.

Hint: For a given h one can write the error probability in terms of a Q-function that depends on $|h|^2$. One can then use the following identity $\mathbb{E}[Q(|h|U)] = \frac{1}{2}\left[1 - \sqrt{\frac{U^2}{U^2+2}}\right]$, for the statistical model described in (8). For example, it can be used in this way: $\mathbb{E}[Q(\sqrt{|h|^2 2\text{SNR}})] = \frac{1}{2}\left[1 - \sqrt{\frac{SNR}{SNR+1}}\right]$.

(d) Use the approximation $\frac{1}{\sqrt{1+x}} \approx 1 - \frac{x}{2}$ for small x, to demonstrate the behavior of the error probability calculated in (d) for large SNR. Using this approximation determine the SNR needed for error probability of 10^{-3}.

Problem 14

(*OFDM System*)
Consider an OFDM transmission scheme in which $N = 4$ subcarriers are used for transmission. The equivalent discrete inter-symbol interference (ISI) channel is modeled by

$$y[k] = x[k] * h[k] + z[k]$$

where $z[k]$ is the additive white Gaussian noise, $x[k]$ is the transmitted signal, $y[k]$ is the received signal, and $h[k]$ is the impulse response of the channel, which has a z-transform of

$$H(z) = 1 + z^{-1} \tag{9}$$

(a) To ensure ISI-free transmission, what is the minimum length of the cyclic prefix?

(b) Suppose we want to transmit the following data packet:

$$1,\ -1,\ 1,\ -1,\ 1,\ 1,\ 1,\ 1$$

using the same length of cyclic prefix(es) as (a). What would be the output sequence of the OFDM modem? You need to write out the generated output sequence and point out the cyclic prefix(es).

(c) Suppose we use the same length of cyclic prefix as part (a), Now, we have learned in class that by using the cyclic prefix of the appropriate length (given in (a) above) we can obtain an equivalent set of parallel channels:

$$\tilde{Y}[m] = \tilde{H}[m]\tilde{S}[m] + \tilde{Z}[m], m = 0, \ldots, N-1,$$

where

$$\tilde{H}[m] = \sum_{l=0}^{L-1} h[l] e^{-j\frac{2\pi m l}{N}}.$$

For the channel given in (9), evaluate explicitly $\tilde{H}[m]$.

Problem 15

(Cyclic Prefix)

Consider a discrete ISI channel described by the model

$$y[x] = s[k] * h[k] + z[k]$$

where $z[k]$ is AWGN noise with variance σ^2, $s[k]$ is the transmitted signal point, $y[k]$ is the received signal, and $h[k]$ is the transfer function of the channel, which has two taps and a z-transform given by

$$H(z) = 1 + 0.9z^{-1}$$

(a) Find the probability of error at the receiver if the inter-symbol interference from h[k] is not corrected, i.e., the inter-symbol interference (ISI) is ignored.

(b) Calculate the energy boost due to the "naive" precoding demonstrated in class, where we suppose

$$s[n] = d[n] - \frac{I[n]}{h_0}$$

so that

$$y[k] = h_0 d[k] + z[k]$$

In other words, find $\mathbb{E}[|s[n]|^2]$ for i.i.d. information symbols compared to the simple scheme given in (a).

(c) Now let use use OFDM transmission. Suppose we use an OFDM block size $N_c = 2$. How long should the cyclic prefix be for the channel given above?

(d) Find the reduction in data rate due to cyclic prefix overhead for $N_c = 2$ and $N_c = 4$. Also find the excess power needed over the trivial scheme of (a).

(e) Find the probability of error if the ISI is corrected using block sizes of $N_c = 2$ and $N_c = 4$. For a fair comparison, the error rate is calculated for the same energy per information bit as used in (a).