Communication Systems EE 132A | UCLA Winter quarter 2014/2015
Prof. Suhas Diggavi | Handout #33

MIDTERM - SOLUTIONS

Wednesday, 11th February, 2015

This exam has 4 problems and 80 points in total.

Instructions

- You are allowed to use <u>1 sheet</u> of paper for reference. No mobile phones or calculators are allowed in the exam.
- You can attempt the problems in any order as long as it is clear as to which problem is being attempted and which solution to the problem you want us to grade.
- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.
- Please solve every problem on **separate paper sheets**.
- It is your responsibility to **number the pages** of your solutions and write on the first sheet the **total number of pages** submitted.

Some relations that might be useful:

- Let $Z = X + Y$, where X and Y are independent random variables with distributions (p.d.f) $f_X(x)$ and $f_Y(y)$, respectively. Then $f_Z(z) = f_X(x) \otimes f_Y(y)$, the convolution of the PDFs of X and Y.
- The energy of a waveform $x(t)$ can be calculated as $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$.
- The energy of a vector representation $\boldsymbol{x}$ of a signal $x(t)$ can be calculated as $E = \boldsymbol{x}^T \boldsymbol{x}$.
- The distance between two constellation points $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ is $d = \|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.
- $Q(a) \triangleq \int_a^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \mathrm{d}x$, the tail probability of a standard normal distribution.
- The distribution of an exponential random variable X with parameter λ is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \text{else.} \end{cases}$$

GOOD LUCK!

Problem 1 (Short Questions)

(a) No. The variance of X is $\frac{\sigma^2}{N}$. The variance of $\sum X_i$ is $N\sigma^2$, and the variance of X is $\frac{1}{N^2}N/\sigma^2$.

A common mistake for this problem is that many of the students did not square the coefficient $1/N$.

(b) No. If you toss an unfair coin, you have a probability of 0 to get a tail.

(c) Yes. To achieve the same P_e, we let $a = b$ according to the union bound. Then we get $E_{a2} < E_{a1}$.

Comment: For the short questions, if your answer is right, but without explanations or with wrong explanations, you will get a deduction of $1 - 3$ points.

Problem 2 (Sufficient Statistics)

(a) According to the equations given in the problem, the conditional probability can be written as

$$f_{Y_1,Y_2|H}(y_1, y_2|0) = f_{Y_1|H}(y_1|0)f_{Y_2|H}(y_2|0) = \frac{1}{2\pi\sigma_1^2}e^{-\frac{(y_1-s)^2+(y_2-s)^2}{2\sigma_1^2}},$$

$$f_{Y_1,Y_2|H}(y_1, y_2|1) = f_{Y_1|H}(y_1|1)f_{Y_2|H}(y_2|1) = \frac{1}{2\pi\sigma_2^2}e^{-\frac{(y_1-s)^2+(y_2-s)^2}{2\sigma_2^2}},$$

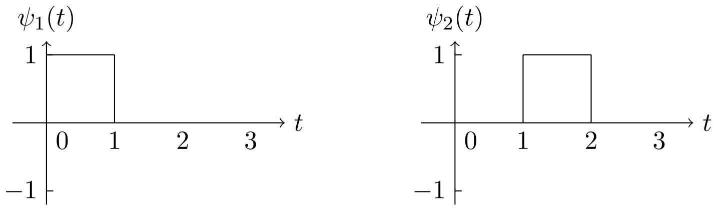
(b) The MAP rule is equivalent to the ML rule, for the hypotheses are with equal probability. Then we have

$$\begin{aligned}
f_{Y_1,Y_2|H}(y_1, y_2|0) &\underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} f_{Y_1,Y_2|H}(y_1, y_2|1) \\
\frac{1}{2\pi\sigma_1^2}e^{-\frac{(y_1-s)^2+(y_2-s)^2}{2\sigma_1^2}} &\underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1}{2\pi\sigma_2^2}e^{-\frac{(y_1-s)^2+(y_2-s)^2}{2\sigma_2^2}} \\
\frac{(y_1-s)^2+(y_2-s)^2}{2\sigma_2^2} - \frac{(y_1-s)^2+(y_2-s)^2}{2\sigma_1^2} &\underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \ln(\frac{\sigma_1^2}{\sigma_2^2}) \\
(y_1 - s)^2 + (y_2 - s)^2 &\underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 4(\ln(\sigma_1) - \ln(\sigma_2))(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_2^2})^{-1}, \quad \text{for } \sigma_1 > \sigma_2 \\
(y_1 - s)^2 + (y_2 - s)^2 &\underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 4(\ln(\sigma_1) - \ln(\sigma_2))(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_2^2})^{-1}, \quad \text{for } \sigma_1 < \sigma_2.
\end{aligned}$$

Comment: A common mistake for this problem is to misusing the N_{ij} in the distribution representation. Since N_{ij} are random variables, the representation containing N_{ij} is seriously wrong, and those who did this would have a score at most 3 for each part. Those who only consider one dimension of the $\boldsymbol{y}$ will have a deduction of $5 - 8$ points.

Problem 3 (Waveforms and Constellation)

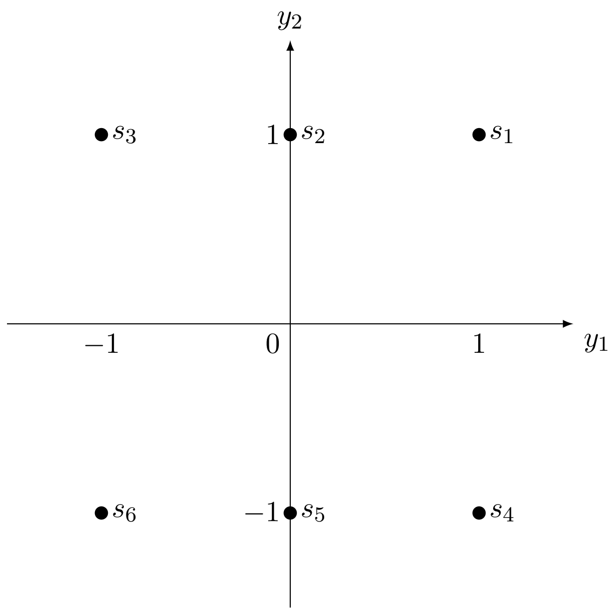
(a) The dimension of this signal set is 2. The orthonormal basis functions are shown as follows:



(b) The coordinates are as follows:

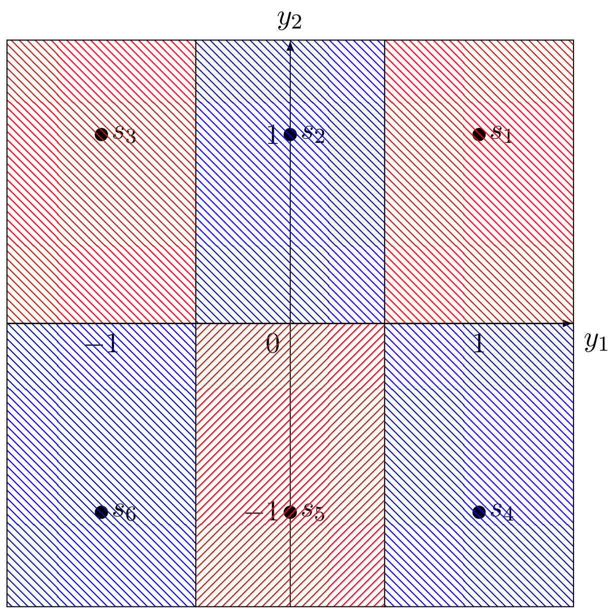
$$s_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad s_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$
$$s_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad s_5 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad s_6 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

The constellation is shown as follows.



(c) From the constellation, we can see that the minimum distance between points is 1. The average energy is $E_a = \frac{1}{6}[4(1^2 + 1^2) + 2(1^2 + 0^2)] = \frac{5}{3}$.

(d) The decision regions are shown in the following figure.



Comment: A common mistake for this problem is that many of the students did not find the orthonormal basis functions correctly. For example, someone may consider $s_1(t)$ and $s_2(t)$ to be orthogonal, but actually they are not. Another common mistake is the calculation of the average energy. You need to square the distance to get the energy. If you did not find the correct basis functions, then you would get 0 points for the first part and corresponding deductions of the following parts.

Problem 4 (QPSK with Phase Error)

(a) In Figure (ii), we can calculate the probability of error for the 4-QAM system with phase error. We first consider the probability of correctness when s_1 is sent:

$$
\begin{aligned}
&P_{c|s_1} = \Pr\{y_1 > 0, y_2 > 0|s_1\} = \Pr\{y_1 > 0|s_1\}\Pr\{y_2 > 0|s_1\}\\
&= [1 - Q(\tfrac{\sqrt{E_p}\cos(\frac{\pi}{4}+\theta)}{\sigma})][1 - Q(\tfrac{\sqrt{E_p}\sin(\frac{\pi}{4}+\theta)}{\sigma})],\\
&= [1 - Q(\tfrac{\sqrt{2E_p}(\cos(\theta)-\sin(\theta))}{2\sigma})][1 - Q(\tfrac{\sqrt{2E_p}(\cos(\theta)+\sin(\theta)}{2\sigma})].
\end{aligned}
$$

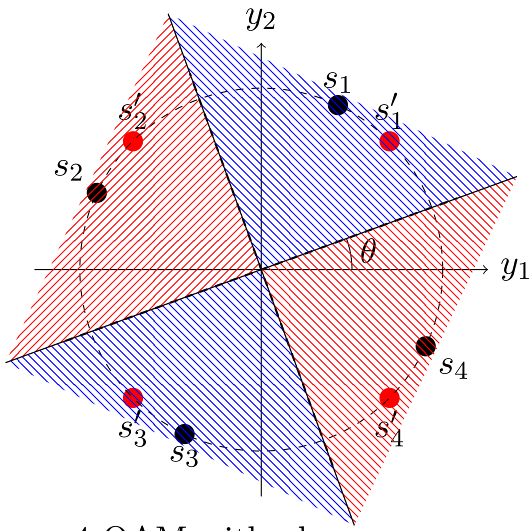
We then calculate the probability of error given that s_1 is sent:

$$
P_{e|s_1} = 1 - P_{c|s_1} = 1 - [1 - Q(\frac{\sqrt{2E_p}(\cos(\theta) - \sin(\theta))}{2\sigma})][1 - Q(\frac{\sqrt{2E_p}(\cos(\theta) + \sin(\theta))}{2\sigma})].
$$

Due to the symmetry, the probabilities of error given each signal is sent are equal. Therefore $P_e = P_{e|s_1}$. Furthermore, since $Q(x)$ is exponentially decreasing in x, we can ignore the second-order term and write the probability of error as

$$
P_e = Q(\frac{\sqrt{2E_p}(\cos(\theta) - \sin(\theta))}{2\sigma}) + Q(\frac{\sqrt{2E_p}(\cos(\theta) + \sin(\theta))}{2\sigma}).
$$

(b) The decision regions are shown in the following figure.



4-QAM with phase error

Comment: A common problem for this problem is the drawing of the decision region. If you drew denote everything clearly, θ, corresponding regions for each point and the axis, you would get a full mark. If you missed something, e.g., θ, or did not clearly draw the regions, axis, etc., you may get a deduction of $1 - 4$ points.