MIDTERM

Wednesday, 11th February, 2015, 10:00-11:50
This exam has 4 problems and 80 points in total.

Instructions

- You are allowed to use 1 sheet of paper for reference. No mobile phones or calculators are allowed in the exam.
- You can attempt the problems in any order as long as it is clear as to which problem is being attempted and which solution to the problem you want us to grade.
- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.
- Please solve every problem on **separate paper sheets**.
- It is your responsibility to **number the pages** of your solutions and write on the first sheet the **total number of pages** submitted.

Some relations that might be useful:

- The energy of a waveform $x(t)$ can be calculated as $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$.
- The energy of a vector representation $\boldsymbol{x}$ of a signal $x(t)$ can be calculated as $E = \boldsymbol{x}^T \boldsymbol{x}$.
- The distance between two constellation points $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ is $d = \|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.
- $Q(a) \triangleq \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x$, the tail probability of a standard normal distribution.
- Triangular identities:
  $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.
  $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$.
  $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.
  $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$.
  $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$.

Good Luck!

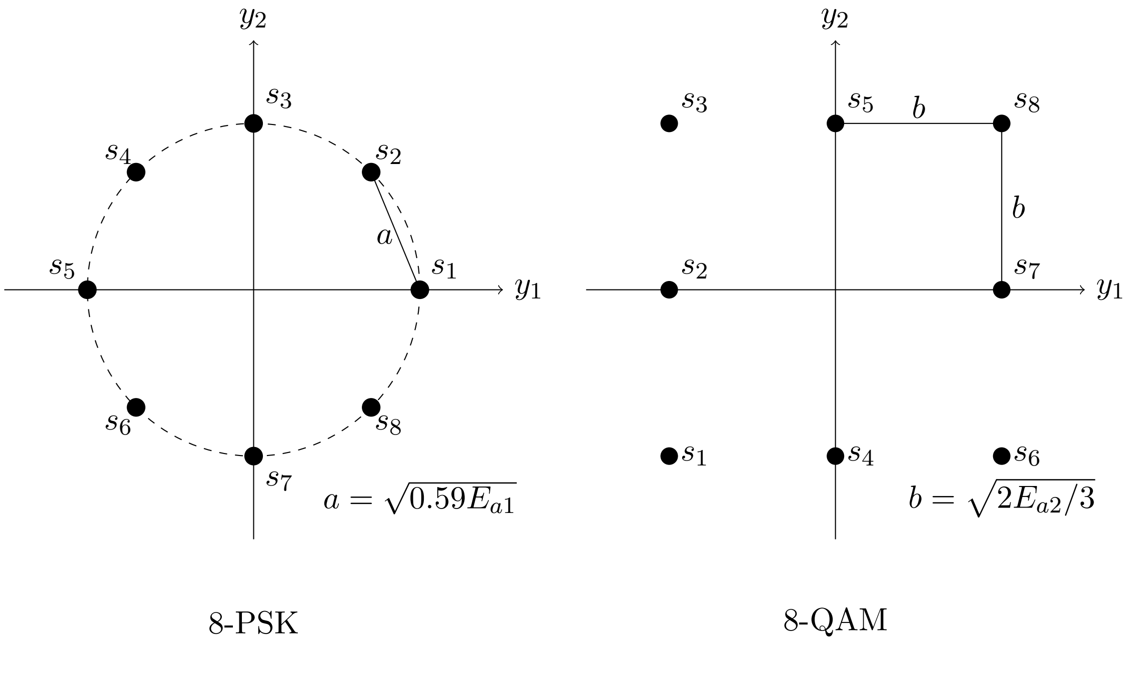
Problem 1 (Short Questions (19 pts))

State whether the following can be labeled with YES or NO. In either case your labeling should be accompanied by a justification. If the statement is NO, demonstrate what would be the correct answer.

(a) Let X_i be i.i.d. Gaussian random variables with mean 0 and variance σ^2. Then $X = \frac{1}{N}\sum_{i=1}^{N} X_i$ is also a Gaussian random variable with mean 0 and variance σ^2. [YES / NO] *[6pts]*

(b) There are three coins, two of which are fair coins and one of which is an unfair coin. When you toss a fair coin, the probability that you get a head is 0.5. When you toss the unfair coin, the probability that you get a head is 1. Now you randomly pick up a coin and want to determine whether it is the unfair coin. To determine this, you have done an experiment: toss the chosen coin thrice and note down the number of heads and tails observed. Then you make an MAP rule-based decision considering only the number of heads and tails observed with no regard to the particular sequence. A result of 2 heads and 1 tail leads to the decision that the chosen coin is unfair. [YES/NO] *[6pts]*

(c) Consider the two communication schemes, 8-PSK and 8-QAM, whose constellations are shown in the following figure. *[7pts]*



E_{a1} and E_{a2} are the average energies of the 8-PSK and 8-QAM schemes. We assume that for both systems, the received discrete signal is (Y_1, Y_2):

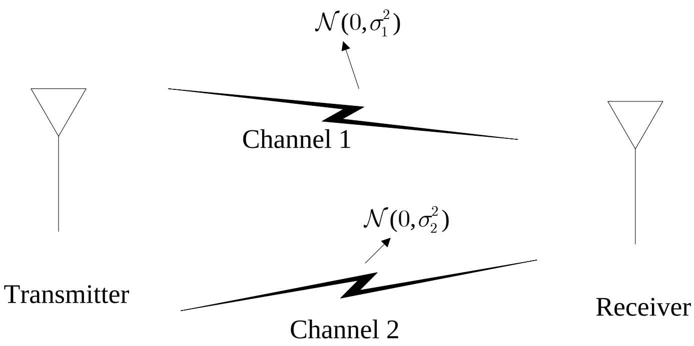
$$Y_1 = X_1 + N_1,$$
$$Y_2 = X_2 + N_2,$$

where N_1 and N_2 are i.i.d. Gaussian noises with mean 0 and variance σ^2. When the m-th signal is transmitted, $(X_1, X_2) = (s_{m1}, s_{m2})$ is the coordinate of the point s_m in the constellation. The AWGN noises in both systems are the same. To achieve the same probability of (symbol) error, the 8-QAM needs a lower average energy to transmit the signals than the 8-PSK. [YES / NO]

(Hint: You can use the union bound $P_e \leq (M-1)Q(\frac{d_{min}}{2\sigma})$ to consider the probability of error, where M is the number of signals, d_{min} is the minimum distance between any two signal points, and σ^2 is the variance of the noise.)

Problem 2 (Sufficient Statistics (20 pts))

Consider a communication system between the transmitter and the receiver, where two channels are possible, as shown in the figure.



The transmitter chooses one channel to transmit a signal twice to the receiver. The receiver knows the signal transmitted by the transmitter, but does not know which channel the transmitter chose. To model this, we assume that the transmitted signal is a fixed signal s, and there are two hypotheses $H = 0$ and $H = 1$, where $H = 0$ corresponds to channel 1 and $H = 1$ corresponds to channel 2. The transmitter chooses the two channels with equal probability. If the transmitter uses channel 1 to transmit the signal, the receiver receives $\boldsymbol{Y} = [Y_1, Y_2]$:

$$Y_1 = s + N_{11},$$
$$Y_2 = s + N_{12},$$

where the noises N_{11} and N_{12} are two i.i.d. Gaussian random variables with mean 0 and variance σ_1^2.

If the transmitter uses channel 2 to transmit the signal, the receiver receives $\boldsymbol{Y} = [Y_1, Y_2]$:

$$Y_1 = s + N_{21},$$
$$Y_2 = s + N_{22},$$

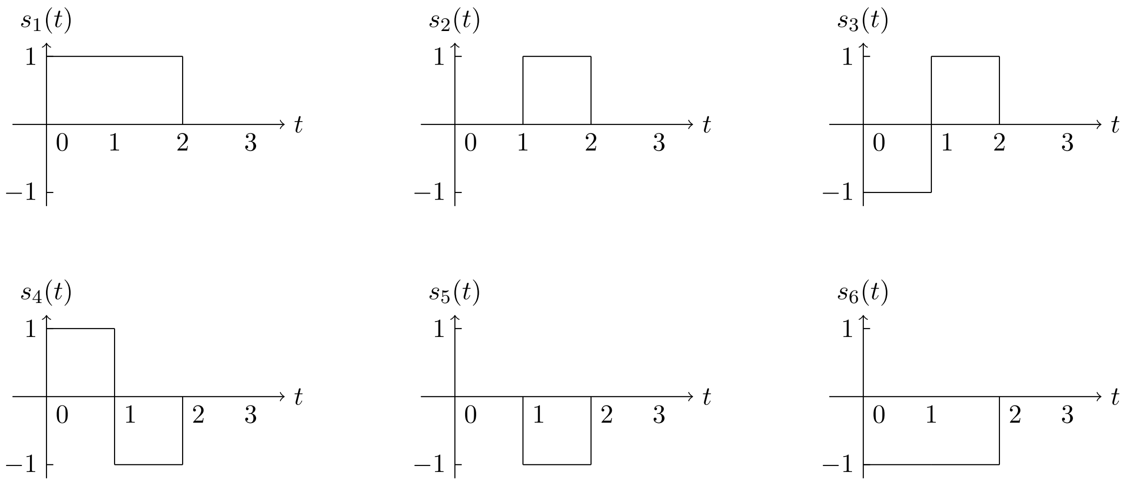
where the noises N_{21} and N_{22} are two i.i.d. Gaussian random variables with mean 0 and variance σ_2^2.

(a) We know that for two independent transmissions using the same channel, $f_{Y_1,Y_2|H}(y_1, y_2|i) = f_{Y_1|H}(y_1|i)f_{Y_2|H}(y_2|i)$, and $f_{Y|H}(y|i) = \frac{1}{\sqrt{2\pi}\sigma_{i+1}} e^{-\frac{(y-s)^2}{2\sigma_{i+1}^2}}$, for $i = 0, 1$. Find the conditional probability density function that the receiver receives a signal $\boldsymbol{y} = [y_1, y_2]$, $f_{Y_1,Y_2|H}(y_1, y_2|0)$, given that channel 1 is used, and the conditional probability density function that the receiver receives a signal $\boldsymbol{y} = [y_1, y_2]$, $f_{Y_1,Y_2|H}(y_1, y_2|1)$, given that channel 2 is used. [*10pts*]

(b) The receiver needs to determine which channel is being used to transmit the signal. Find the MAP decision rule for the receiver. [*10pts*]

Problem 3 (Waveforms and Constellation (18 pts))

Consider the signal waveforms in the following figure. Let the number of messages $M = 6$ and if $H = i, i = 1, 2, \cdots, 6$, we transmit the signal $s_i(t)$. We assume that the signals are transmitted with equal probability.



(a) What is the dimension of this signal set? Find a set of orthonormal basis functions for this signal set. (Hint: You can do this without using the Gram-Schmidt procedure.) *[4pts]*

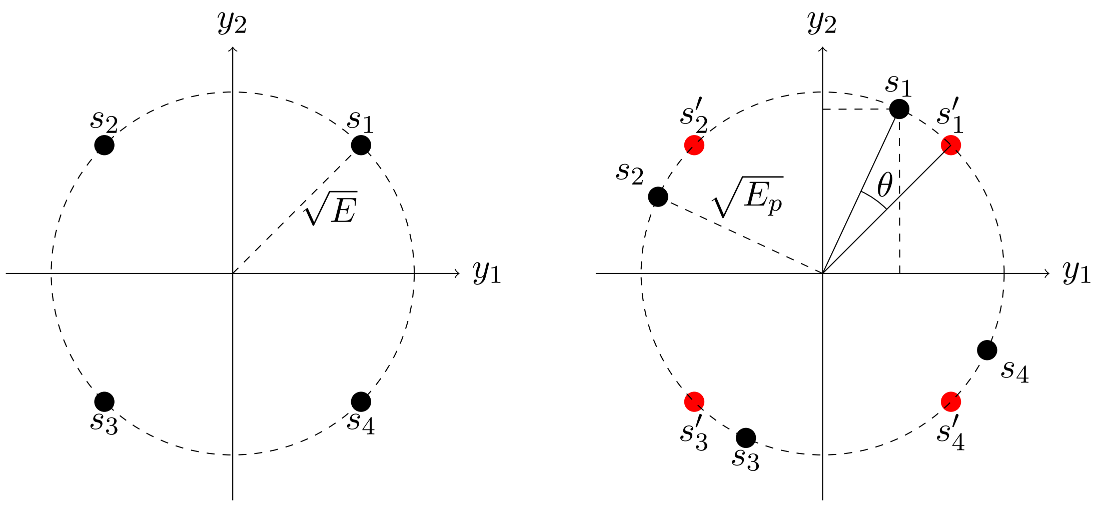
(b) Find the coordinates of the six data symbols in the coordinate system based on the basis functions you found in (a) and draw the constellation points in the coordinate system. *[4pts]*

(c) What is the minimum distance between two constellation points? What is the average energy used to transmit the signals? *[6pts]*

(d) Draw the optimal decision region of each constellation point in the same constellation of Part (b). *[4pts]*

Problem 4 (QPSK with Phase Error (23 pts))

We consider the 4-QAM (or QPSK) system with the constellation shown in Figure (i).



(i) 4-QAM　　(ii) 4-QAM with phase error

For the 4-QAM system, we assume that the four constellation points are transmitted with equal probability. However, in real systems, there may be a phase error for the transmitted signals, as shown in Figure (ii). In this case, the real constellation points s_i $(i = 1, 2, 3, 4)$ are rotated an angle θ $(< \pi/4)$ from these ideal constellation points s'_i $(i = 1, 2, 3, 4)$.

The receiver does not know the phase error, and considers the constellation points as the ideal ones. Based on these ideal constellation points s'_i $(i = 1, 2, 3, 4)$, the receiver determines its MAP decision rule, called rule-1.

Suppose that the average energy of the 4-QAM system is E; the average energy of the 4-QAM system with phase error is E_p, and the zero-mean Gaussian noise has variance σ^2.

(a) In Figure (i), we can calculate the probability of error for the 4-QAM system. We first consider the probability of correctness when s_1 is sent: *[15pts]*

$$P_{c|s_1} = \Pr\{y_1 > 0, y_2 > 0|s_1\} = \Pr\{y_1 > 0|s_1\} \Pr\{y_2 > 0|s_1\} = [1 - Q(\sqrt{\frac{E}{2\sigma^2}})]^2,$$

where the received signals y_1 and y_2 are independent Gaussians given s_1 is sent. We then calculate the probability of error given that s_1 is sent:

$$P_{e|s_1} = 1 - P_{c|s_1} = 1 - [1 - Q(\sqrt{\frac{E}{2\sigma^2}})]^2.$$

Due to the symmetry, the probabilities of error given each signal is sent are equal. Therefore $P_e = 1 - [1 - Q(\sqrt{\frac{E}{2\sigma^2}})]^2 = 2Q(\sqrt{\frac{E}{2\sigma^2}}) - [Q(\sqrt{\frac{E}{2\sigma^2}})]^2$. Furthermore, since $Q(x)$ is exponentially decreasing in x and $[Q(x)]^2$ is a small term compared with $Q(x)$, we can ignore the second-order term $[Q(\sqrt{\frac{E}{2\sigma^2}})]^2$ and write $P_e = 2Q(\sqrt{\frac{E}{2\sigma^2}})$.

Following the above idea, calculate the probability of error in terms of E_p, θ and σ for the 4-QAM system with phase error in Figure (ii). In Figure (ii), the real constellation points are s_i $(i = 1, 2, 3, 4)$, but the receiver does not know the phase error and makes decisions based on rule-1, which is designed for the ideal points.

(Hint: Note that the rule-1 may not be optimal for the 4-QAM system with phase error. Consider the coordinates of s_1 and find the probability of correctness first.)

(b) If the receiver knows this phase error θ, it will change its optimal MAP decision rule to rule-2. Draw the optimal decision region for the receiver based on rule-2 in Figure (ii). *[8pts]*