Communication Systems EE 132A
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UCLA Winter quarter 2013/2014
Handout #34

MIDTERM - SOLUTIONS

Thursday, 13th February, 2014

This exam has 4 problems and 80 points in total.

Instructions

- You are allowed to use 1 sheet of paper for reference. No mobile phones or calculators are allowed in the exam.
- You can attempt the problems in any order as long as it is clear as to which problem is being attempted and which solution to the problem you want us to grade.
- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.
- Please solve every problem on **separate paper sheets**.
- It is your responsibility to **number the pages** of your solutions and write on the first sheet the **total number of pages** submitted.

Some relations that might be useful:

- Let $Z = X + Y$, where X and Y are independent random variables with distributions (p.d.f) $f_X(x)$ and $f_Y(y)$, respectively. Then $f_Z(z) = f_X(x) \otimes f_Y(y)$, the convolution of the PDFs of X and Y.
- The energy of a waveform $x(t)$ can be calculated as $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$.
- The energy of a vector representation $\boldsymbol{x}$ of a signal $x(t)$ can be calculated as $E = \boldsymbol{x}^T \boldsymbol{x}$.
- The distance between two constellation points $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ is $d = \|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.
- $Q(a) \triangleq \int_a^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \mathrm{d}x$, the tail probability of a standard normal distribution.
- The distribution of an exponential random variable X with parameter λ is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \text{else.} \end{cases}$$

GOOD LUCK!

Problem 1 (Short Questions (16 pts))

State whether the following can be labeled with YES or NO. In either case your labeling should be accompanied by a justification. If the statement is NO, demonstrate what would be the correct answer.

(a) Let A and B be two i.i.d. Gaussian random variables with mean μ and variance σ^2. Then $\mathbb{P}[2A + B > 3] = Q\left(\frac{\sqrt{3}(1-\mu)}{\sigma}\right)$ [YES / NO] *[5pts]*

$\rightarrow$ NO. Let $Y = 2A + B$. Y is a Gaussian random variable since linear transformation of a Gaussian random vector results in another Gaussian random vector. Y is Gaussian with mean 3μ and variance $5\sigma^2$. Hence,

$$\begin{aligned}\mathbb{P}[2A + B > 3] &= \mathbb{P}[Y > 3] && (1)\\ &= \mathbb{P}\left[\frac{Y - 3\mu}{\sqrt{5}\sigma} > \frac{3 - 3\mu}{\sqrt{5}\sigma}\right]\\ &= Q\left(\frac{3 - 3\mu}{\sqrt{5}\sigma}\right). && (2)\end{aligned}$$

(b) For a QPSK constellation, with symbols taken from $\{(1, 0), (0, 1), (-1, 0), (0, -1)\}$), and a channel that adds a phase error that is uniformly distributed on $[0, 2\pi]$, the minimal error probability that can be achieved is $\frac{3}{4}$. [YES / NO] *[4pts]*

$\rightarrow$ YES (assuming equiprobable signaling; other answers shall be accepted with valid reasoning and computation since the problem does not specify the input probability mass function to be used; for example one can argue that transmitting only two points from the constellation each with probability $\frac{1}{2}$ will lead to a minimal probability of error of $\frac{1}{2}$ but such a scheme would not be QPSK anymore. The most degenerate case would be transmitting only one point with probability 1 but such a scheme doesn't even require any communication in the first place).

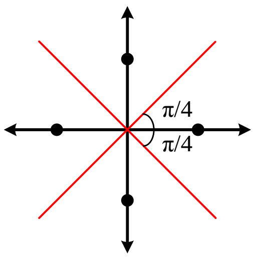


Figure 1: Constellation with ML regions, assuming equiprobable signaling

Minimal probability of error requires the decision rule to be the Maximum a-posterior probability (MAP) decision rule. For equiprobable signaling, MAP rule reduces to the Maximum likelihood (ML) decoding rule, which, for 2-D constellations, can be sketched. The total probability of error can be computed as the probability of error conditioned on any point, without loss of generality, since all the points have the same geometry to their decision regions (refer Fig. 1).

$$
\begin{aligned}
\mathbb{P}[\text{error}] =& \mathbb{P}[\text{error} \mid \text{any transmitted point}] && (3)\\
=& 1 - \mathbb{P}\left[-\frac{\pi}{4} \leq \text{Phase noise} \leq \frac{\pi}{4}\right]\\
=& 1 - \frac{1}{4}\\
=& \frac{3}{4}. && (4)
\end{aligned}
$$

(c) There are two coins of identical appearance on a table. You know one of the coins is a fair coin, meaning $\mathbb{P}[\text{Heads}] = \mathbb{P}[\text{Tails}] = 0.5$, and the other is biased, with $\mathbb{P}[\text{Heads}] = 0.75$, and $\mathbb{P}[\text{Tails} = 0.25]$, but you don't know which one is the fair coin. You pick up one coin and toss it N times. The result of the i^{th} toss is described by a random variable $X_i = x_i$, where $x_i = 1$ for a head and 0 for a tail. The quantity $S_N = \sum_{i=1}^{N} x_i$ is a sufficient statistic for determining whether the coin you tossed is unbiased. [YES / NO] *[7pts]*

→ YES, S_N is a sufficient statistic. To prove this, it needs to be shown that, if the bias is indicated as θ,

$$\mathbb{P}[(X_1, X_2, \ldots, X_N) \mid S_N, \theta] = \mathbb{P}[(X_1, X_2, \ldots, X_N) \mid S_N], \tag{5}$$

which is seen as following. If S_N is known, then since X_i are i.i.d. with the same parameter,

$$\mathbb{P}[(X_1, X_2, \ldots, X_N) \mid S_N] = \begin{cases} \frac{1}{\binom{N}{S_N}} & \sum_i X_i = S_N, \\ 0 & \text{else.} \end{cases} \tag{6}$$

The above is independent of θ, thereby proving that S_N is a sufficient statistic.

OR

It needs to be shown that Fisher-Neyman factorization theorem holds with S_N as a sufficient statistic (this proof follows more in the lines of the course). The following characterization of the conditional distributions explain this. Let $H = 0$ denote the prior event that the coin is biased and $H = 1$ denote the prior event that is otherwise.

$$P_{\boldsymbol{X}|H}(\boldsymbol{X} \mid 0) = \left(\frac{1}{2}\right)^{S_N} \left(\frac{1}{2}\right)^{n-S_N} \tag{7}$$

$$P_{\boldsymbol{X}|H}(\boldsymbol{X} \mid 1) = \left(\frac{3}{4}\right)^{S_N} \left(\frac{1}{4}\right)^{n-S_N}. \tag{8}$$

where the conditional distributions $P_{\boldsymbol{X}|H}$ are factorized into $1 \times g_H(T(\boldsymbol{X}))$ where $T(\boldsymbol{X}) = S_N$ is the sufficient characteristic. Since the factorization of conditional distributions as

$$P_{\boldsymbol{X},\theta} = h(\boldsymbol{X}) \times g_\theta(T(\boldsymbol{X}) = S_N), \quad \theta = 0.5, 0.75$$

holds with $h(\boldsymbol{X}) = 1$, S_N is a sufficient statistic for θ. Note that this statistic is independent of N and does not require asymptoticity.

Problem 2 (Bank Lines (23 pts))

Upon walking into a bank, you see two queues of equal number of people, each being served by a teller. You model the time it takes the tellers to serve a client by exponential distributions with different parameters λ_1 and λ_2, where $\lambda_1 > \lambda_2$. However, you don't know which teller serves people at the rate λ_1 and which teller serves at the rate of λ_2. You observe random service times $\mathbf{X} = (X_1, X_2)$, where X_1, X_2 are the times it takes Teller A to service her first and second customer, respectively. Similarly, you observe $\mathbf{Y} = (Y_1, Y_2)$, the random service times for Teller B's first two customers. Assume X_1, X_2, Y_1 and Y_2 are all independent of each other. Your objective is to use your observations to determine, with the least probability of error, which teller serves at the faster rate of λ_1 and join that queue. *Note:* A teller serving faster has a higher "rate" of service λ.

(a) Let $H = 0$ denote the case in which Teller A is faster, and $H = 1$ denote the case in which Teller B is faster. Find the 2 joint conditional PDFs $f_{\mathbf{X},\mathbf{Y}|H}(\mathbf{x}, \mathbf{y} \mid 0)$ and $f_{\mathbf{X},\mathbf{Y}|H}(\mathbf{x}, \mathbf{y} \mid 1)$ for this hypothesis testing problem. *[9pts]*

→ When $H = 0$, Teller A's service times are exponential with parameter λ_1, and Teller B's service times are exponential with parameter λ_2. Since X_1, X_2, Y_1, Y_2 are mutually independent random variables, the joint conditional pdf is simply the product of each marginal conditional pdf:

$$\begin{aligned} f_{\mathbf{X},\mathbf{Y}|H}(\mathbf{x}, \mathbf{y} \mid 0) &= (\lambda_1 e^{-\lambda_1 x_1})(\lambda_1 e^{-\lambda_1 x_2})(\lambda_2 e^{-\lambda_2 y_1})(\lambda_2 e^{-\lambda_2 y_2}) \\ &= \lambda_1^2 \lambda_2^2 e^{-\lambda_1(x_1+x_2)-\lambda_2(y_1+y_2)}, \quad x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0. \end{aligned} \tag{9}$$

Similiarly,

$$f_{\mathbf{X},\mathbf{Y}|H}(\mathbf{x}, \mathbf{y} \mid 1) = \lambda_1^2 \lambda_2^2 e^{-\lambda_2(x_1+x_2)-\lambda_1(y_1+y_2)}, \quad x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0. \tag{10}$$

(b) Find the optimum Maximum Likelihood (ML) decision rule for this hypothesis testing problem. *[6pts]*

→ The ML rule is given by

$$f_{\mathbf{X},\mathbf{Y}|H}(\mathbf{x}, \mathbf{y} \mid 0) \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} f_{\mathbf{X},\mathbf{Y}|H}(\mathbf{x}, \mathbf{y} \mid 1). \tag{11}$$

We immediately see that the $\lambda_1^2 \lambda_2^2$ terms on both sides cancel, and we can take the log of both sides to give:

$$-\lambda_1(x_1 + x_2) - \lambda_2(y_1 + y_2) \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} -\lambda_2(x_1 + x_2) - \lambda_1(y_1 + y_2),$$

which simplifies further to

$$x_1 + x_2 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} y_1 + y_2. \tag{12}$$

Note that the inequalities change direction as the way it is written with x_1, x_2 on the left hand side because gathering $\lambda_2 - \lambda_1$ together yields a negative number on both sides as $\lambda_1 > \lambda_2$.

(c) Compute the error probability assuming that $P_H(H_0) = P_H(H_1) = \frac{1}{2}$. *[8pts]*

$\rightarrow$ Begin by defining $U = X_1 + X_2$ and $V = Y_1 + Y_2$. Then,

$$
\begin{aligned}
P_e &= \frac{1}{2}\left[\mathbb{P}(\hat{H} = 1 \mid H = 0) + \mathbb{P}(\hat{H} = 0 \mid H = 1)\right] \\
&= \frac{1}{2}\left[\mathbb{P}(U > V \mid H = 0) + \mathbb{P}(V > U \mid H = 1)\right]. \qquad (13)
\end{aligned}
$$

To find these conditional error probabilities, we first must find the conditional distributions of U and V. Since X_1 and X_2 are independent, the conditional pdf of U given H_0 can be found by convolving the conditional pdfs of X_1 and X_2 given H:

$$
\begin{aligned}
f_{U|H}(u \mid 0) &= \int_{-\infty}^{\infty} f_{X_1|H}(x_1 \mid 0) f_{X_2|H}(u - x_1 \mid 0) dx_1 \\
&= \int_{-\infty}^{\infty} \lambda_1^2 e^{-\lambda_1(x_1 + (u - x_1))} dx_1 \\
&= \lambda_1^2 \int_0^z e^{-\lambda_1 u} dx_1 \\
&= \lambda_1^2 u e^{-\lambda_1 u}, \quad u \geq 0. \qquad (14)
\end{aligned}
$$

We can find the remaining three conditional pdf's in a similar manner:

$$
\begin{aligned}
f_{U|H}(u \mid 1) &= \lambda_2^2 u e^{-\lambda_2 u}, \quad u \geq 0, \\
f_{V|H}(v \mid 0) &= \lambda_2^2 v e^{-\lambda_2 v}, \quad v \geq 0, \\
f_{V|H}(v \mid 1) &= \lambda_1^2 v e^{-\lambda_1 v}, \quad v \geq 0.
\end{aligned}
$$

Now, we find the conditional probabilities:

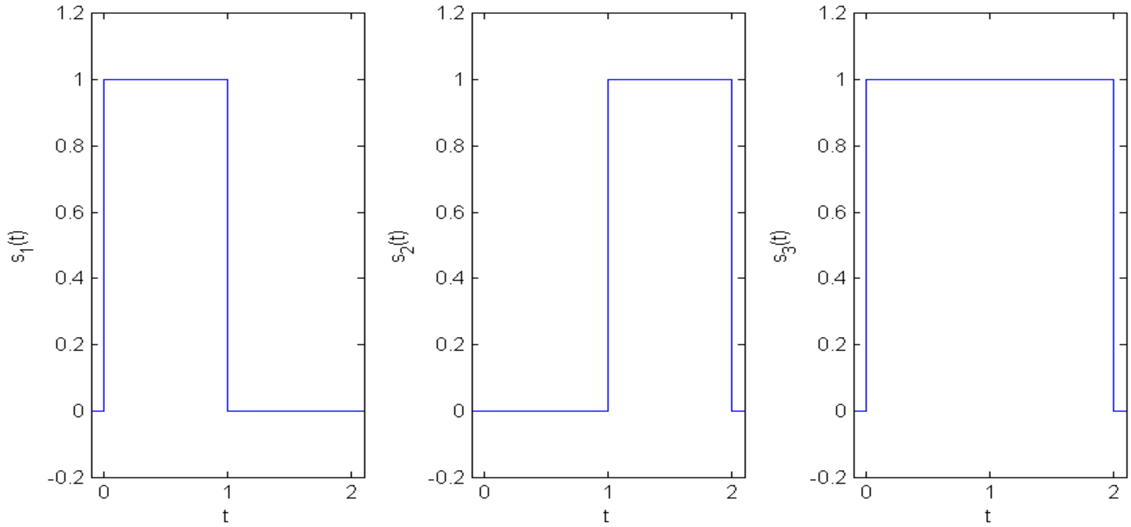
$$
\begin{aligned}
\mathbb{P}[V > U \mid H = 1] &= \int_0^{\infty} \int_u^{\infty} \lambda_1^2 \lambda_2^2 u v e^{-\lambda_2 u} e^{-\lambda_1 v} dv\ du \qquad (15) \\
&= \lambda_1^2 \lambda_2^2 \int_0^{\infty} u e^{-\lambda_2 u} \left(\int_u^{\infty} v e^{-\lambda_1 v} dv \right) du \\
&= \lambda_2^2 \int_0^{\infty} u e^{-(\lambda_1 + \lambda_2) u} (1 + \lambda_1 u)\ du \\
&= \left(\lambda_2^2 \int_0^{\infty} u e^{-(\lambda_1 + \lambda_2) u}\ du \right) + \left(\lambda_2^2 \lambda_1 \int_0^{\infty} u^2 e^{-(\lambda_1 + \lambda_2) u}\ du \right) \\
&= \frac{\lambda_2^2}{(\lambda_1 + \lambda_2)^2} \left(1 + \frac{2\lambda_1}{\lambda_1 + \lambda_2} \right) \\
&= \frac{\lambda_2^2 (3\lambda_1 + \lambda_2)}{(\lambda_1 + \lambda_2)^3}. \qquad (16)
\end{aligned}
$$

By inspection, we can also see that the integral for the other conditional error term will be the same, resulting in an average error probability of

$$
P_e = \frac{\lambda_2^2 (3\lambda_1 + \lambda_2)}{(\lambda_1 + \lambda_2)^3}. \qquad (17)
$$

Problem 3 (Waveform Representations(**17 pts**))

Consider the following signals. Let the number of messages $M = 3$ and if $H = i, i = 1, 2, 3$, we transmit the signal $s_i(t)$:



(a) Find a set of orthonormal basis functions for this signal set. *[3pts]*

$\rightarrow$ The norm of the three signals is computed and found out to be:

$$\|s_1(t)\| = 1, \quad (18)$$
$$\|s_2(t)\| = 1, \quad (19)$$
$$\|s_3(t)\| = \sqrt{2}. \quad (20)$$

Going through Gram-Schmidt would yield $s_1(t)$ and $s_2(t)$ as two orthonormal basis functions for the signaling set. $s_1(t)$ is orthogonal to $s_2(t)$ since $\langle s_1(t), s_2(t)\rangle = \int_0^1 1 \cdot 0 \text{ dt} + \int_1^2 0 \cdot 1 \text{ dt} = 0$.

(b) Find the data symbols corresponding to the signals above for the basis functions you found in (a). *[3pts]*

$\rightarrow$ The data symbols corresponding to the signals using the above standard basis are:

$$\mathbf{s}_1 = [1, 0], \quad (21)$$
$$\mathbf{s}_2 = [0, 1], \quad (22)$$
$$\mathbf{s}_3 = [1, 1]. \quad (23)$$

(c) What is the average transmitted energy if all messages are equiprobable? *[3pts]*

$\rightarrow$ The average energy of this signaling set is found out to be

$$\mathcal{E} = \frac{1}{3}[1 + 1 + 2] = \frac{4}{3}. \quad (24)$$

(d) Now suppose we send the signal over an AWGN channel, *i.e.,* *[8pts]*

$$H = i : y(t) = s_i(t) + N(t),$$

where $N(t)$ is AWGN noise with power spectral density of $\frac{N_0}{2} = 4 \times 10^{-4}$ Watts/Hz. Further suppose that all messages are equally likely, *i.e.*, $P_H(1) = P_H(2) = P_H(3) = \frac{1}{3}$. Bound the error probability for this communication scheme using the most appropriate form of the union bound.

→ The constellation, along with the ML decision regions, is sketched as shown in Fig. 2. The most appropriate (tightest) bound on the probability of error that can be obtained

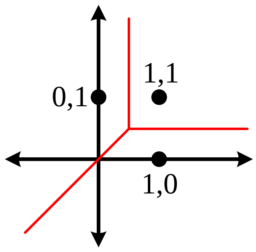


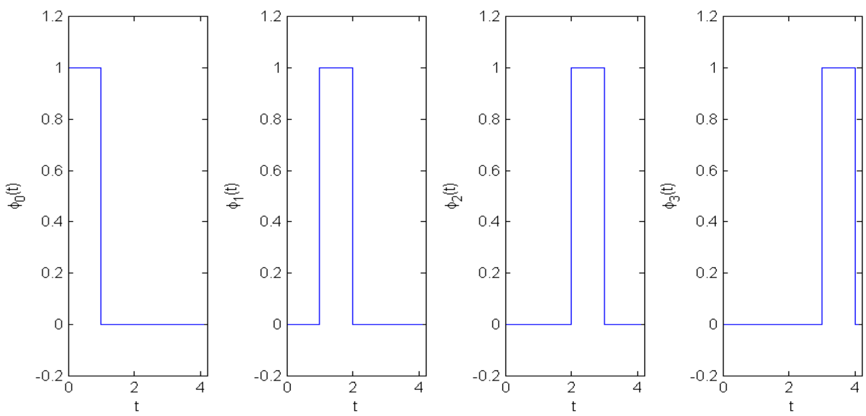
Figure 2: Constellation with ML regions

is the individual nearest neighbour union bound which would yield

$$\mathrm{P}_{e,\mathrm{INNUB}} \leq \frac{2}{3} Q\left(\frac{1}{2\sigma}\right) + \frac{2}{3}\left[Q\left(\frac{1}{2\sigma}\right) + Q\left(\frac{1}{\sqrt{2}\sigma}\right)\right]. \tag{25}$$

Problem 4 (Constellation Trade-offs (24 pts))

Suppose you wish to design a system in which 2 bits are transmitted per symbol. Your basis functions are rectangular pulses of width 1 allowing you to use up to 4 time "slots", as shown.



Your primary goal is to create a signaling scheme in which, given an average total energy E and AWGN channel noise power (variance) of σ^2, the probability of error is at a minimum. A secondary goal is to complete each 2-bit transmission using the fewest number of time slots. You can assume that the system signals equiprobably.

Now consider the following signaling schemes:

	Scheme 1	**Scheme 2**	**Scheme 3**
$\boldsymbol{x}_0$	$\left(\sqrt{E}, 0, 0, 0\right)$	$\left(\sqrt{E}, 0, 0, 0\right)$	$\left(\sqrt{\frac{E}{2}}, 0, 0, 0\right)$
$\boldsymbol{x}_1$	$\left(0, \sqrt{E}, 0, 0\right)$	$\left(-\sqrt{E}, 0, 0, 0\right)$	$\left(-\sqrt{\frac{E}{2}}, 0, 0, 0\right)$
$\boldsymbol{x}_2$	$\left(0, 0, \sqrt{E}, 0\right)$	$\left(0, \sqrt{E}, 0, 0\right)$	$\left(\sqrt{\frac{3E}{2}}, 0, 0, 0\right)$
$\boldsymbol{x}_3$	$\left(0, 0, 0, \sqrt{E}\right)$	$\left(0, -\sqrt{E}, 0, 0\right)$	$\left(-\sqrt{\frac{3E}{2}}, 0, 0, 0\right)$

Your communication system transmitting waveforms for each of the 2 bit messages, per symbol, can be represented as:

$$
\begin{aligned}
H = i, i \in [0, 3] : y(t) =& s_i(t) + Z(t), \\
s_i(t) =& \sum_{j=0}^{3} x_{i,j}\phi_j(t), \\
\boldsymbol{x}_i =& [x_{i,0}, x_{i,1}, x_{i,2}, x_{i,3}]^T.
\end{aligned}
$$

(a) Which of the scheme(s) have the largest minimum distance between transmitted signal points? [*9pts*]

$\rightarrow$ Scheme 1 has a minimum distance of $\sqrt{2E}$ (the same distance between all points), Scheme 2 has a minimum distance (say, between points $\left(-\sqrt{E}, 0, 0, 0\right)$ and $\left(0, \sqrt{E}, 0, 0\right)$) of $\sqrt{2E}$ and Scheme 3 has a minimum distance (say, between points $\sqrt{\frac{3E}{2}}$ and $\sqrt{\frac{E}{2}}$) of less than $\sqrt{2E}$. Schemes 1 and 2 are better than the third in terms of minimum distance and they have the "largest" minimum distance.

(b) Of the scheme(s) with the largest minimum distance, which one requires the fewest number of time slots to complete a transmission? [*3pts*]

Note: A time slot is "used" by the system only if there's at least one signaling waveform that has a non-zero voltage value in that time-slot.

$\rightarrow$ Of the two schemes identified above, Scheme 2 requires the fewest number of time slots to complete a transmission. It should also be noted that Scheme 2 is better than Scheme 1 even considering only distances as a measure since Scheme 2 has two distances that are of magnitude $2\sqrt{E}$ while Scheme 1 has the same distance of $\sqrt{2E}$ between any two points. Aside, Scheme 1 is an example of "orthogonal" signaling whereas Scheme 2 is an example of "Bi-orthogonal" signaling.

(c) Given your answer in part (b), derive the MAP rule for the system and calculate the probability of error as a function of $\sqrt{E}$ and σ^2 for that symbol constellation. [*12pts*]

$\rightarrow$ *Decision rule:*

The constellation in Scheme 2 is an example of a Bi-orthogonal signaling scheme with this scheme representing a QPSK system. It has been mentioned that the system signals equiprobably. Hence the ML decision regions are enough to make the best decision on a received $\mathbf{Y} = [y_1, y_2]^T$. The constellation with ML regions is given as follows:

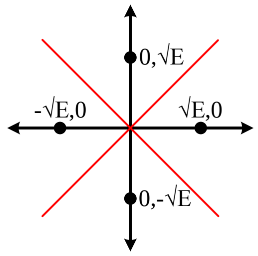


Figure 3: QPSK with ML regions

The decision rule is written out as:

$$d(\mathbf{Y}) = \begin{cases} \boldsymbol{x}_0 & \{y_1 > y_2\} \cap \{y_1 > -y_2\}, \\ \boldsymbol{x}_2 & -y_2 \leq y_1 \leq y_2, \\ \boldsymbol{x}_1 & \{y_1 < y_2\} \cap \{y_1 < -y_2\}, \\ \boldsymbol{x}_3 & \text{else.} \end{cases} \tag{26}$$

Probability of error:

To find the probability of error over an AWGN channel, rotate the constellation 45°, as shown in Fig. 4. This rotation is allowed since AWGN is symmetric and the probability of error is rotation invariant. The probability of error is found out, by symmetry, conditioning

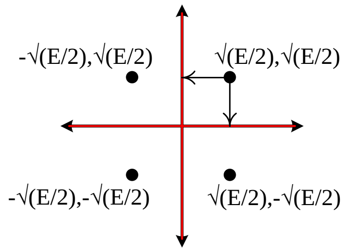


Figure 4: QPSK with rotated ML regions

on the point $\tilde{\boldsymbol{x}}_0 = \left(\sqrt{\frac{E}{2}}, \sqrt{\frac{E}{2}}\right)$. Let the two noise components be N_1, N_2.

$$\begin{aligned} \mathrm{P}_e = \mathrm{P}_{e|\tilde{\boldsymbol{x}}_0} =& 1 - \mathrm{P}_{c|\tilde{\boldsymbol{x}}_0} \qquad (27) \\ =& 1 - \mathbb{P}\left[N_1 < \sqrt{\frac{E}{2}}\right] \times \mathbb{P}\left[N_2 < \sqrt{\frac{E}{2}}\right] \\ =& 1 - \left[1 - Q\left(\sqrt{\frac{E}{2\sigma^2}}\right)\right]^2 \\ =& 2Q\left(\sqrt{\frac{E}{2\sigma^2}}\right) - \left[Q\left(\sqrt{\frac{E}{2\sigma^2}}\right)\right]^2. \qquad (28) \end{aligned}$$

Aside: An individual nearest neighbour union bound would yield

$$\mathrm{P}_{e,\mathrm{INNUB}} \leq 2Q\left(\sqrt{\frac{E}{2\sigma^2}}\right) \tag{29}$$

which is very close to the accurate value obtained from (28) if the argument inside the Q-function is large enough since the quadratic term in (28) will vanish exponentially with increasing argument inside the Q-function. The argument in this discussion is nothing but the SNR which is given by $\frac{E}{2\sigma^2}$ where E is the average energy of the constellation. Note that for a 2-D signaling scheme such as this, the noise power is $2\sigma^2$ with the variance equally distributed as σ^2 along each dimension.