MIDTERM

Wednesday, 12th February, 2014, 10:00-11:50
This exam has 4 problems and 80 points in total.

Instructions

- You are allowed to use 1 sheet of paper for reference. No mobile phones or calculators are allowed in the exam.
- You can attempt the problems in any order as long as it is clear as to which problem is being attempted and which solution to the problem you want us to grade.
- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.
- Please solve every problem on **separate paper sheets**.
- It is your responsibility to **number the pages** of your solutions and write on the first sheet the **total number of pages** submitted.

Some relations that might be useful:

- Let $Z = X + Y$, where X and Y are independent random variables with distributions (p.d.f) $f_X(x)$ and $f_Y(y)$, respectively. Then $f_Z(z) = f_X(x) \otimes f_Y(y)$, the convolution of the PDFs of X and Y.
- The energy of a waveform $x(t)$ can be calculated as $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$.
- The energy of a vector representation $\boldsymbol{x}$ of a signal $x(t)$ can be calculated as $E = \boldsymbol{x}^T \boldsymbol{x}$.
- The distance between two constellation points $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ is $d = \|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.
- $Q(a) \triangleq \int_a^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \mathrm{d}x$, the tail probability of a standard normal distribution.
- The distribution of an exponential random variable X with parameter λ is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \text{else.} \end{cases}$$

GOOD LUCK!

Problem 1 (Short Questions (16 pts))

State whether the following can be labeled with YES or NO. In either case your labeling should be accompanied by a justification. If the statement is NO, demonstrate what would be the correct answer.

(a) Let A and B be two i.i.d. Gaussian random variables with mean μ and variance σ^2. Then $\mathbb{P}[2A + B > 3] = Q\left(\frac{\sqrt{3}(1-\mu)}{\sigma}\right)$ [YES / NO] *[5pts]*

(b) For a QPSK constellation, with symbols taken from $\{(1, 0), (0, 1), (-1, 0), (0, -1)\}$), and a channel that adds a phase error that is uniformly distributed on $[0, 2\pi]$, the minimal error probability that can be achieved is $\frac{3}{4}$. [YES / NO] *[4pts]*

(c) There are two coins of identical appearance on a table. You know one of the coins is a fair coin, meaning $\mathbb{P}[\text{Heads}] = \mathbb{P}[\text{Tails}] = 0.5$, and the other is biased, with $\mathbb{P}[\text{Heads}] = 0.75$, and $\mathbb{P}[\text{Tails} = 0.25]$, but you don't know which one is the fair coin. You pick up one coin and toss it N times. The result of the i^{th} toss is described by a random variable $X_i = x_i$, where $x_i = 1$ for a head and 0 for a tail. The quantity $S_N = \sum_{i=1}^{N} x_i$ is a sufficient statistic for determining whether the coin you tossed is unbiased. [YES / NO] *[7pts]*

Problem 2 (Bank Lines (23 pts))

Upon walking into a bank, you see two queues of equal number of people, each being served by a teller. You model the time it takes the tellers to serve a client by exponential distributions with different parameters λ_1 and λ_2, where $\lambda_1 > \lambda_2$. However, you don't know which teller serves people at the rate λ_1 and which teller serves at the rate of λ_2. You observe random service times $\mathbf{X} = (X_1, X_2)$, where X_1, X_2 are the times it takes Teller A to service her first and second customer, respectively. Similarly, you observe $\mathbf{Y} = (Y_1, Y_2)$, the random service times for Teller B's first two customers. Assume X_1, X_2, Y_1 and Y_2 are all independent of each other. Your objective is to use your observations to determine, with the least probability of error, which teller serves at the faster rate of λ_1 and join that queue. *Note:* A teller serving faster has a higher "rate" of service λ.

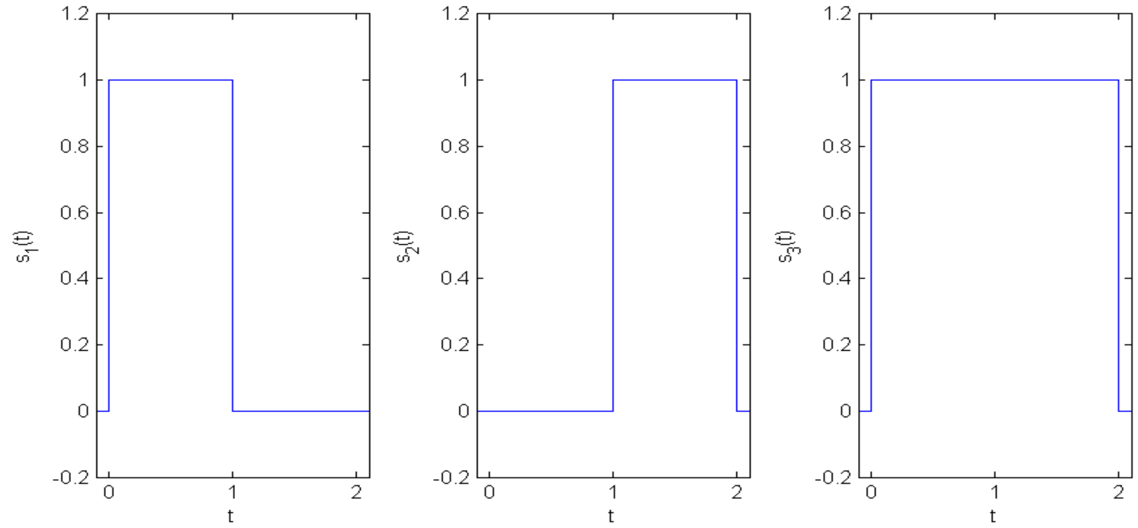
(a) Let H_0 denote the case in which Teller A is faster, and H_1 denote the case in which Teller B is faster. Find the 2 joint conditional PDFs $f_{\mathbf{X},\mathbf{Y}|\mathbf{H}}(\mathbf{x}, \mathbf{y}|0)$ and $f_{\mathbf{X},\mathbf{Y}|\mathbf{H}}(\mathbf{x}, \mathbf{y}|1)$ for this hypothesis testing problem. *[9pts]*

(b) Find the optimum Maximum Likelihood (ML) decision rule for this hypothesis testing problem. *[6pts]*

(c) Compute the error probability assuming that $P_H(H_0) = P_H(H_1) = \frac{1}{2}$. *[8pts]*

Problem 3 (Waveform Representations(17 pts))

Consider the following signals. Let the number of messages $M = 3$ and if $H = i, i = 1, 2, 3$, we transmit the signal $s_i(t)$:



(a) Find a set of orthonormal basis functions for this signal set. [*3pts*]

(b) Find the data symbols corresponding to the signals above for the basis functions you found in (a). [*3pts*]

(c) What is the average transmitted energy if all messages are equiprobable? [*3pts*]

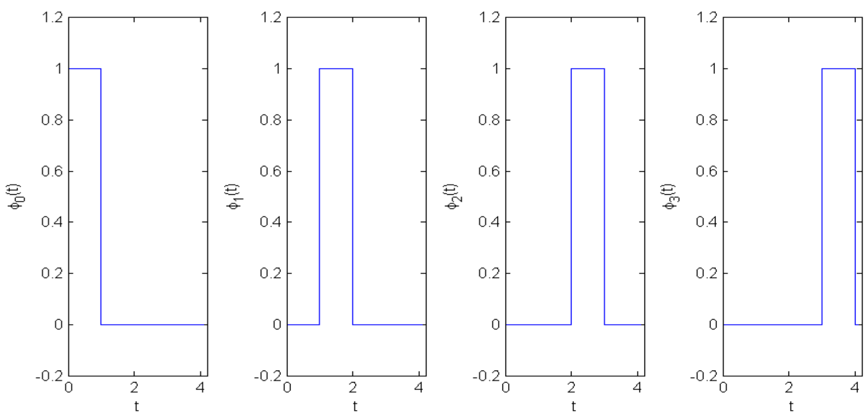
(d) Now suppose we send the signal over an AWGN channel, *i.e.*, [*8pts*]

$$H = i : y(t) = s_i(t) + N(t),$$

where $N(t)$ is AWGN noise with power spectral density of $\frac{N_0}{2} = 4 \times 10^{-4}$ Watts/Hz. Further suppose that all messages are equally likely, *i.e.*, $P_H(1) = P_H(2) = P_H(3) = \frac{1}{3}$. Bound the error probability for this communication scheme using the most appropriate form of the union bound.

Problem 4 (Constellation Trade-offs (24 pts))

Suppose you wish to design a system in which 2 bits are transmitted per symbol. Your basis functions are rectangular pulses of width 1 allowing you to use up to 4 time "slots", as shown below:



Your primary goal is to create a signaling scheme in which, given an average total energy E and AWGN channel noise power (variance) of σ^2, the probability of error is at a minimum. A secondary goal is to complete each 2-bit transmission using the fewest number of time slots. You can assume that the system signals equiprobably.

Now consider the following signaling schemes:

	Scheme 1	**Scheme 2**	**Scheme 3**
$\boldsymbol{x}_0$	$\left(\sqrt{E}, 0, 0, 0\right)$	$\left(\sqrt{E}, 0, 0, 0\right)$	$\left(\sqrt{\frac{E}{2}}, 0, 0, 0\right)$
$\boldsymbol{x}_1$	$\left(0, \sqrt{E}, 0, 0\right)$	$\left(-\sqrt{E}, 0, 0, 0\right)$	$\left(-\sqrt{\frac{E}{2}}, 0, 0, 0\right)$
$\boldsymbol{x}_2$	$\left(0, 0, \sqrt{E}, 0\right)$	$\left(0, \sqrt{E}, 0, 0\right)$	$\left(\sqrt{\frac{3E}{2}}, 0, 0, 0\right)$
$\boldsymbol{x}_3$	$\left(0, 0, 0, \sqrt{E}\right)$	$\left(0, -\sqrt{E}, 0, 0\right)$	$\left(-\sqrt{\frac{3E}{2}}, 0, 0, 0\right)$

Your communication system transmitting waveforms for each of the 2 bit messages, per symbol, can be represented as:

$$\begin{aligned} H = i, i \in [0,3] : y(t) =& s_i(t) + Z(t), \\ s_i(t) =& \sum_{j=0}^{3} x_{i,j}\phi_j(t), \\ \boldsymbol{x}_i =& [x_{i,0}, x_{i,1}, x_{i,2}, x_{i,3}]^T. \end{aligned}$$

(a) Which of the scheme(s) have the largest minimum distance between transmitted signal points? [*9pts*]

(b) Of the scheme(s) with the largest minimum distance, which one requires the fewest number of time slots to complete a transmission? [*3pts*]

Note: A time slot is “used” by the system only if there’s at least one signaling waveform that has a non-zero voltage value in that time-slot.

(c) Given your answer in part (b), derive the MAP rule for the system and calculate the probability of error as a function of $\sqrt{E}$ and σ^2 for that symbol constellation. [*12pts*]