MIDTERM SOLUTIONS

Problem 1 (Short questions (20 pts))

State whether the statements can be labeled with YES or NO. In either case your labeling should be accompanied by a justification. If the statement is NO, demonstrate what would be the correct answer.

(a) If (X_1, X_2) are jointly Gaussian random variables with a covariance matrix $\Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$ then the variance of $X_1 + X_2$ is 4. [YES / NO] *[5pts]*

(b) For orthogonal signalling with equiprobable symbols $p_H(0) = p_H(1) = \frac{1}{2}$ which results in constellations $\mathbf{s}_0 = \begin{bmatrix} \sqrt{E} \\ 0 \end{bmatrix}$ and $\mathbf{s}_1 = \begin{bmatrix} 0 \\ \sqrt{E} \end{bmatrix}$. This constellation has the lowest power for the same error probability and rate. [YES / NO] *[3pts]*

(c) Consider binary messages ($M = 2$) transmitted over a AWGN waveform channel, *i.e.,* $R(t) = x(t) + N(t)$, where $x(t) = s_i(t)$ if $H = i \in \{0, 1\}$, where $s_0(t), s_1(t)$ are *not* orthogonal. Then $\int R(t)s_0(t)dt$ and $\int R(t)s_1(t)dt$ form a set of sufficient statistics. [YES / NO] *[4pts]*

(d) Assume that we have a binary hypothesis testing problem with observation in $\mathbb{R}^n$. Let $\mathcal{R}_0$ and $\mathcal{R}_1$ be the MAP decision regions where $\mathcal{R}_0 \cup \mathcal{R}_1 = \mathbb{R}^n$. If we enlarge the region $\mathcal{R}_1$ then (increasing here means *non-decreasing*, i.e., constant or strictly increasing):

(i) The average error probability increases. [YES / NO] *[3pts]*

(ii) The conditional error probability under H_1 increases. [YES / NO] *[5pts]*

Solution:

(a) No. The variance of $X_1 + X_2$ is given by:

$$\begin{aligned} \text{var}(X_1 + X_2) &= \mathbb{E}\left\{(X_1 + X_2 - \mathbb{E}X_1 - \mathbb{E}X_2)^2\right\} \\ &= \mathbb{E}\left\{(X_1 - \mathbb{E}X_1)^2\right\} + \mathbb{E}\left\{(X_2 - \mathbb{E}X_2)^2\right\} + 2\mathbb{E}\left\{(X_1 - \mathbb{E}X_1)(X_2 - \mathbb{E}X_2)\right\} \\ &= 1 + 1 + 2 \times 0.5 \\ &= 3 \end{aligned}$$

(b) No. This constellation is not centered around origin. Let $\boldsymbol{v} = \begin{bmatrix} \sqrt{\frac{E}{2}} \\ \sqrt{\frac{E}{2}} \end{bmatrix}$. Then the one that has the lowest power for the same probability of error is given by $\tilde{\boldsymbol{s}}_0 = \boldsymbol{s}_0 - \boldsymbol{v}$ and $\tilde{\boldsymbol{s}}_1 = \boldsymbol{s}_1 - \boldsymbol{v}$.

(c) Yes. If $s_0(t)$ and $s_1(t)$ are linearly dependent, then the dimension of the space spanned by $s_0(t)$ and $s_1(t)$ is one. The orthonormal basis for $\{s_0(t), s_1(t)\}$ is either $\phi(t) = s_0(t)/\|s_0(t)\|$ or $\phi(t) = s_1(t)/\|s_1(t)\|$. Since $\int R(t)\phi(t)dt$ is the sufficient statistics, either $\int R(t)s_0(t)dt$ or $\int R(t)s_1(t)dt$ is also the sufficient statistics because $s_0(t)$ and $s_1(t)$ are just scaled versions of $\phi(t)$. On the other hand, if $s_0(t)$ and $s_1(t)$ are linearly independent, then the dimension of the space spanned by $s_0(t)$ and $s_1(t)$ is two. Let $\phi_1(t)$ and $\phi_2(t)$ be the orthonormal basis functions of $\{s_0(t), s_1(t)\}$ so that

$$s_0(t) = s_{01}\phi_1(t) + s_{02}\phi_2(t)$$
$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t)$$

The vector representations $\boldsymbol{s}_0 = \begin{bmatrix} s_{01} \\ s_{02} \end{bmatrix}$ and $\boldsymbol{s}_1 = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}$ are linearly indepdendent as well, which means we can solve $\int R(t)\phi_1(t)dt$ and $\int R(t)\phi_2(t)dt$ uniquely from the following equations:

$$\int R(t)s_0(t)dt = s_{01}\int R(t)\phi_1(t)dt + s_{02}\int R(t)\phi_2(t)dt$$
$$\int R(t)s_1(t)dt = s_{11}\int R(t)\phi_1(t)dt + s_{12}\int R(t)\phi_2(t)dt$$

Thus, $\int R(t)s_0(t)dt$ and $\int R(t)s_1(t)dt$ are the sufficient statistics.

(d) (i) Yes. Since MAP rule is optimal in minimizing the probability of error, adjusting the decision boundary will always increase the probability of error.

(ii) No. The conditional probability of error given H_1 will decrease. This is because this conditional probability of error is given by the following integral:

$$\Pr(\text{error}|H_1) = \int_{Y\in\mathcal{R}_0} f_{Y|H}(y|H_1)dy$$

When we enlarge the region $\mathcal{R}_1$, the region $\mathcal{R}_0$ will decrease and thus the above integral will decrease.

Problem 2 (Noisy pulse length (17 pts))

For binary *equiprobable* message transmission, we have the following channel when we transmit a message $H = 0$ the receiver gets a pulse of random length ℓ which is distributed uniformly between $[0, Q]$. When we transmit a message $H = 1$, the receiver gets a pulse which is distributed exponentially as $\exp(-\ell)$. More formally, we get a pulse of random length ℓ distributed as,

$$f_L(\ell) = \begin{cases} U[0,Q] & \text{if } H = 0 \\ \exp(-\ell),\ \ell \geq 0 & \text{if } H = 1 \end{cases}$$

(a) Derive the optimal decoder for this channel and draw the decision regions. *[5pts]*
Hint: Consider cases $Q \leq 1$ and $Q > 1$.

(b) Find the error probability for this optimal decoder. *[8pts]*

(c) For what value of $Q \leq 1$ is $P_e < \frac{1}{2}e^{-4}$? *[4pts]*

Solution:

(a) We can write out the conditional pdf for H_0 and H_1, respectively:

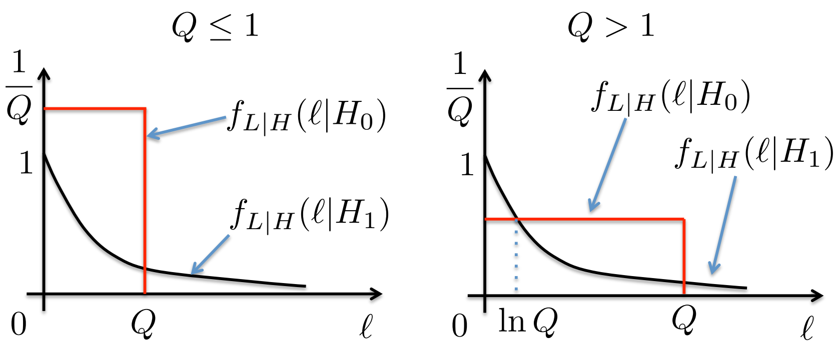
$$H_0: \; f_{L|H}(\ell|H_0) = \begin{cases} \frac{1}{Q} & \ell \in [0, Q] \\ 0 & \text{otherwise} \end{cases}$$

$$H_1: \; f_{L|H}(\ell|H_1) = \begin{cases} e^{-\ell} & \ell \in [0, Q] \\ 0 & \text{otherwise} \end{cases}$$

The MAP rule is given by

$$f_{L|H}(\ell|H_1) \underset{H_0}{\overset{H_1}{\gtrless}} f_{L|H}(\ell|H_0)$$

In order to solve the above inequality for the decision regions, we need to discuss the problem in two cases: $Q \leq 1$ and $Q > 1$ (see the figure below). Therefore, we have



1. $\underline{Q \leq 1}$:

$$\begin{aligned} H_0: &\quad [0, Q] \\ H_1: &\quad [Q, \infty) \end{aligned}$$

2. $\underline{Q > 1}$:

$$\begin{aligned} H_0: &\quad [\ln Q, Q] \\ H_1: &\quad [0, \ln Q) \cup [Q\infty) \end{aligned}$$

(b) We need to compute the probability of error in two cases: $Q \leq 1$ and $Q > 1$.

1. $\underline{Q \leq 1}$:

$$\begin{aligned} P_e &= \Pr\{H_0\} \int_Q^\infty f_{L|H}(\ell|H_0) d\ell + \Pr\{H_1\} \int_0^Q f_{L|H}(\ell|H_1) d\ell \\ &= \frac{1}{2} \times 0 + \frac{1}{2} \int_0^Q f_{L|H}(\ell|H_1) d\ell \\ &= \frac{1}{2} \int_0^Q e^{-\ell} d\ell \\ &= \frac{1}{2}(1 - e^{-Q}) \end{aligned}$$

2. $\underline{Q > 1}$:

$$P_e = \Pr\{H_0\}\left[\int_0^{\ln Q} f_{L|H}(\ell|H_0)d\ell + \int_Q^{\infty} f_{L|H}(\ell|H_0)d\ell\right] + \Pr\{H_1\}\int_{\ln Q}^{Q} f_{L|H}(\ell|H_1)d\ell$$
$$= \Pr\{H_0\}\int_0^{\ln Q} f_{L|H}(\ell|H_0)d\ell + \Pr\{H_1\}\int_{\ln Q}^{Q} f_{L|H}(\ell|H_1)d\ell$$
$$= \frac{1}{2}\int_0^{\ln Q}\frac{1}{Q}d\ell + \frac{1}{2}\int_{\ln Q}^{Q} e^{-\ell}d\ell$$
$$= \frac{1}{2}\left[\frac{\ln Q + 1}{Q} - e^{-Q}\right]$$

(c) For $Q \leq 1$, the probability of error is given by $P_e = \frac{1}{2}(1 - e^{-Q})$. To ensure $P_e < \frac{1}{2}e^{-4}$, it suffices to have

$$\frac{1}{2}(1 - e^{-Q}) < \frac{1}{2}e^{-4}$$

from which we get $Q < \ln \frac{1}{1-e^{-4}}$.

Problem 3 (Waveform representations(**17 pts**))

Consider the following signals:

$$s_0(t) = \begin{cases} \sqrt{\frac{2}{T}}\cos(\frac{2\pi t}{T} + \frac{\pi}{6}) & \text{if } t \in [0, T] \\ 0 & \text{else} \end{cases}$$
$$s_1(t) = \begin{cases} \sqrt{\frac{2}{T}}\cos(\frac{2\pi t}{T} + \frac{5\pi}{6}) & \text{if } t \in [0, T] \\ 0 & \text{else} \end{cases}$$
$$s_2(t) = \begin{cases} \sqrt{\frac{2}{T}}\cos(\frac{2\pi t}{T} + \frac{3\pi}{2}) & \text{if } t \in [0, T] \\ 0 & \text{else} \end{cases}$$

(a) Find a set of orthonormal basis functions for this signal set. *[3pts]*
Hint: Use the identity for $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.

(b) Find the data symbols corresponding to the signals above for the basis functions you found in (a). *[3pts]*

(c) What is the average energy of transmitted if all messages are equiprobable? *[2pts]*

(d) Now suppose we send the signal over a AWGN channel, *i.e.*, *[7pts]*

$$H = i : y(t) = s_i(t) + N(t),$$

where $N(t)$ is AWGN noise with power spectral density of $\frac{N_0}{2} = 3 \times 10^{-2}$Watts/Hz. Further suppose that all messages are equally likely, *i.e.*, $p_H(0) = \frac{1}{3} = p_H(1) = p_H(2)$. Bound the error probability for this communication scheme using the most appropriate form of the union bound.
Hint: Use the Q-function tables given in the Instructions.

(e) Can we reduce the error probability for the same energy by tranforming (translating) this signaling scheme? If so, demonstrate the the optimal translation to minimize error probability. (you do not need to calculate the this error probability but just give the new transformed waveforms). [2pts]

Solution:

(a) The orthonormal basis functions for this signal set is

$$\phi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}\cos(\frac{2\pi t}{T}) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2(t) = \begin{cases} \sqrt{\frac{2}{T}}\sin(\frac{2\pi t}{T}) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

We can verify that they are orthonormal.

(b) Using the identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, we have

$$\sqrt{\frac{2}{T}}\cos(\frac{2\pi t}{T} + \frac{\pi}{6}) = \cos(\frac{\pi}{6})\sqrt{\frac{2}{T}}\cos(\frac{2\pi t}{T}) - \sin(\frac{\pi}{6})\sqrt{\frac{2}{T}}\sin(\frac{2\pi t}{T})$$

Therefore

$$s_0(t) = \cos(\frac{\pi}{6})\phi_1(t) - \sin(\frac{\pi}{6})\phi_2(t)$$

Likewise, for $s_1(t)$ and $s_2(t)$, we have

$$s_1(t) = \cos(\frac{5\pi}{6})\phi_1(t) - \sin(\frac{5\pi}{6})\phi_2(t)$$
$$s_2(t) = \cos(\frac{3\pi}{2})\phi_1(t) - \sin(\frac{3\pi}{2})\phi_2(t)$$

Thus, we have the vector representations for the three signals as:

$$\boldsymbol{s}_0 = \begin{bmatrix} \cos(\frac{\pi}{6}) \\ -\sin(\frac{\pi}{6}) \end{bmatrix} \quad \boldsymbol{s}_1 = \begin{bmatrix} \cos(\frac{5\pi}{6}) \\ -\sin(\frac{5\pi}{6}) \end{bmatrix} \quad \boldsymbol{s}_2 = \begin{bmatrix} \cos(\frac{3\pi}{2}) \\ -\sin(\frac{3\pi}{2}) \end{bmatrix}$$

(c) Note that $\|\boldsymbol{s}_0\|^2 = \|\boldsymbol{s}_1\|^2 = \|\boldsymbol{s}_2\|^2 = 1$. Therefore, the average energy is $E_{\text{avg}} = 1$.

(d) The three points have equal distance to each other, which is $d_{\min} = \sqrt{3}$. The number of nearest neighbors is $N = 2$. Therefore,

$$P_e \le 2Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) = 2Q(5) \approx 5.74 \times 10^{-7}$$

(e) No. The constellation is already centered around the origin.

Problem 4 (Repetition signaling (26 pts))

Consider the following repetition signaling schemes.
Case 1: If we want to transmit binary message, represented as 1 bit, we do the following: To transmit 0, we send the vector $\mathbf{X} = (-\sqrt{E}, -\sqrt{E}, \dots, -\sqrt{E})$ of length $2n$, where $\sqrt{E}$ is a given amplitude. To transmit 1, we send the vector $\mathbf{X} = (\sqrt{E}, \sqrt{E}, \dots, \sqrt{E})$.
Case 2: Suppose we want to transmit one of four messages represented by 2 bits as follows. We represent message $H = 0$ by 00, $H = 1$ by 01, $H = 2$ by 10 and $H = 3$ by 11. In short, message i is represented as two bits $b_{i,1}, b_{i,2}$. To transmit we do the following: we encode message i (equivalent to two bits $b_{i,1}, b_{i,2}$) as $\mathbf{s}_i = \begin{bmatrix} s_{i,1} \\ s_{i,2} \end{bmatrix}$, where

$$s_{i,k} = \begin{cases} -\sqrt{E} & \text{if } b_{i,k} = 0 \\ \sqrt{E} & \text{if } b_{i,k} = 1 \end{cases}$$

To send message $H = i \in \{0, 1, 2, 3\}$, we send the vector $\mathbf{X} = (\mathbf{s}_i^T, \mathbf{s}_i^T, \dots, \mathbf{s}_i^T)$ of length $2n$, where $\mathbf{s}_i^T = \begin{bmatrix} s_{i,1} & s_{i,2} \end{bmatrix}$.

The received signal is $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$, where $\mathbf{Z} = (Z_1, Z_2, \dots, Z_n)$ is the additive noise of the channel consisting of iid components distributed like $\eta(0, \sigma^2 = 1)$. Assume that all messages are equiprobable.

(a) Derive the MAP rule and find the error probability as a function of n, $\sqrt{E}$, and σ for Case 1. *[4pts]*

(b) Derive the MAP rule and find the error probability as a function of n, $\sqrt{E}$, and σ for Case 2. *[10pts]*

(c) Suppose that the noise of the channel increases to $\sigma^2 = 2$. Assume that n is fixed but we can change E. Call the new value E'. How shall we set E' so that the error probability stays the same. Do this for both Case 1 and Case 2. *[3pts]*

(d) Suppose that the noise of the channel increases to $\sigma^2 = 2$. Assume that we can not change E but we can increase n. Call the new value n'. How shall we choose n' so that the error probability stays the same. Repeat this for both cases. *[3pts]*

(e) Suppose that we are now given a fixed budget for energy per bit transmitted. Which transmission scheme would you choose to get lower *message* error probability? *[6pts]*

Solution:

(a) For Case 1, the orthonormal basis vector is

$$\boldsymbol{\phi} = \frac{1}{\sqrt{2n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Note that for H_1, the transmitted signal is $\boldsymbol{X} = \sqrt{2nE} \cdot \boldsymbol{\phi}$ and for H_0, the transmitted signal is $\boldsymbol{X} = -\sqrt{2nE} \cdot \boldsymbol{\phi}$. Thus, the vector representation for $\boldsymbol{s}_1$ and $\boldsymbol{s}_0$ in this case are:

$$\boldsymbol{s}_1 = \sqrt{2nE}$$
$$\boldsymbol{s}_0 = -\sqrt{2nE}$$

The MAP rule is

$$\|\boldsymbol{Y} - \boldsymbol{s}_1\|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \|\boldsymbol{Y} - \boldsymbol{s}_0\|^2$$

Noting that $\|\boldsymbol{s}_0\|^2 = \|\boldsymbol{s}_1\|^2 = 2nE$, we can reduce the above MAP rule into

$$\boldsymbol{Y}^T \boldsymbol{\phi} \underset{H_0}{\overset{H_1}{\gtrless}} 0 \quad \Leftrightarrow \quad \frac{1}{\sqrt{2n}}(Y_1 + \cdots + Y_{2n}) \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

The distance between these two points are $d_{\min} = 2\sqrt{2nE}$. Hence, the probability of error is given by

$$P_e = Q\left(\sqrt{\frac{d_{\min}^2}{4\sigma^2}}\right) = Q\left(\sqrt{\frac{2nE}{\sigma^2}}\right)$$

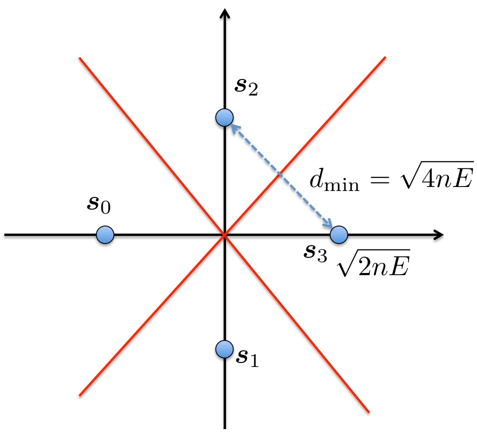
(b) For Case 2, we can choose the following two vectors as the orthonormal basis:

$$\boldsymbol{\phi}_1 = \frac{1}{\sqrt{2n}}\begin{bmatrix}1\\1\\\vdots\\1\\1\end{bmatrix} \quad \boldsymbol{\phi}_2 = \frac{1}{\sqrt{2n}}\begin{bmatrix}1\\-1\\\vdots\\1\\-1\end{bmatrix}$$

Then, under $H_0, \ldots, H_3$, the transmitted signals can be expressed as

$$\begin{aligned} H_0 :& \boldsymbol{X} = -\sqrt{2nE}\boldsymbol{\phi}_1 \\ H_1 :& \boldsymbol{X} = -\sqrt{2nE}\boldsymbol{\phi}_2 \\ H_2 :& \boldsymbol{X} = \sqrt{2nE}\boldsymbol{\phi}_2 \\ H_3 :& \boldsymbol{X} = \sqrt{2nE}\boldsymbol{\phi}_1 \end{aligned}$$

The constellation and the MAP decision boundaries are shown in the following figure.



The probability of error can be expressed as

$$\begin{aligned} P_e &= 1 - \left[1 - Q\left(\sqrt{\frac{d_{\min}^2}{4\sigma^2}}\right)\right]^2 \\ &= 1 - \left[1 - Q\left(\sqrt{\frac{nE}{\sigma^2}}\right)\right]^2 \\ &= 2Q\left(\sqrt{\frac{nE}{\sigma^2}}\right) - \left[Q\left(\sqrt{\frac{nE}{\sigma^2}}\right)\right]^2 \\ &\leq 2Q\left(\sqrt{\frac{nE}{\sigma^2}}\right) \end{aligned}$$

(c) For both cases, if the noise power σ^2 increases from 1 to 2, we need to increase the energy of the signal from E to $2E$, namely, $E' = 2E$.

(d) For both cases, if the noise power σ^2 increases from 1 to 2, we need to set $n' = 2n$. $E' = 2E$.

(e) For Case 1 (1−bit scheme), the energy per bit is $E_b = 2nE$. Therefore,

$$P_e = Q\left(\sqrt{\frac{E_b}{\sigma^2}}\right)$$

For Case 2 (2−bit scheme), the energy per bit is $E_b = 2nE/log_2 4 = nE$. Therefore,

$$P_e = 2Q\left(\sqrt{\frac{E_b}{\sigma^2}}\right) - \left[Q\left(\sqrt{\frac{E_b}{\sigma^2}}\right)\right]^2 \approx 2Q\left(\sqrt{\frac{E_b}{\sigma^2}}\right)$$

Therefore, we prefer 1−bit scheme if we want to ensure the same probability of message error given the same E_b.
