MIDTERM

Thursday, 16th February, 2012, 16:00-17:50
This exam has 4 problems and 80 points in total.

Instructions

- You are allowed to use 1 sheet of paper for reference. No mobile phones or calculators are allowed in the exam.
- You can attempt the problems in any order as long as it is clear which problem is being attempted and which solution to the problem you want us to grade.
- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.
- Please solve every problem on **separate paper sheets**.
- It is your responsibility to **number the pages** of your solutions and write on the first sheet the **total number of pages** submitted.

Integrals:

$$\int_0^\infty e^{-at}e^{-j2\pi ft}dt = \frac{1}{a+j2\pi f}$$
$$\int_{-\infty}^0 e^{at}e^{-j2\pi ft}dt = \frac{1}{a-j2\pi f}$$

Q-function Values:

$Q(0) = 0.5$	$Q(4) = 3.17\text{e-}5$
$Q(0.5) = 0.31$	$Q(4.5) = 3.4\text{e-}6$
$Q(2/3) = 0.25$	$Q(5) = 2.87\text{e-}7$
$Q(1) = 0.16$	$Q(7.5) = 3.22\text{e-}14$
$Q(1.5) = 0.067$	$Q(10) = 7.77\text{e-}24$
$Q(2) = 0.023$	$Q(12.5) = 3.86\text{e-}36$
$Q(2.5) = 0.0062$	$Q(15) = 3.85\text{e-}51$
$Q(3) = 0.0013$	$Q(25) = 3.38\text{e-}138$
$Q(3.5) = 0.00023$	$Q(\infty) = 0$

GOOD LUCK!

Problem 1 (Detection in uniform noise (20 pts))

A one-dimensional additive noise channel, $Y = X + Z$, has uniform noise distribution (see also Figure 1):

$$p_Z(z) = \begin{cases} \frac{1}{L} & |z| \leq \frac{L}{2} \\ 0 & |z| > \frac{L}{2} \end{cases}$$

where $\frac{L}{2}$ is the maximum noise magnitude. The input X has binary antipodal constellation with equally likely input values $X = \pm 1$, *i.e.,* if $M = 0, X = 1$ and if $M = 1, X = -1$; see also Figure 2. The noise Z is independent of X.

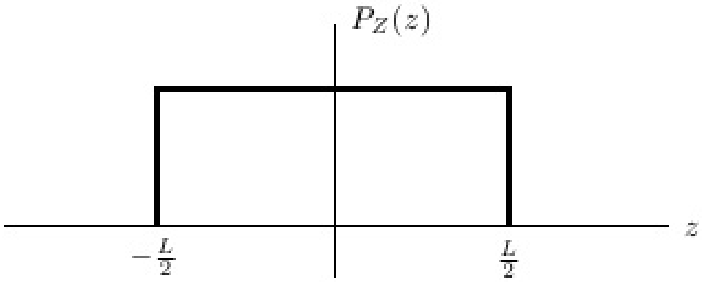


Figure 1: Uniform noise probability density function

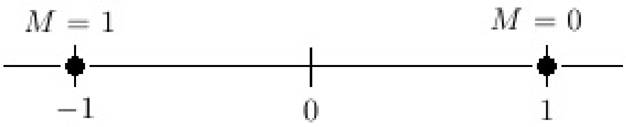


Figure 2: Signal constellation for transmission

(a) Design an optimum detector (showing decision regions is sufficient.) *[5pts]*

(b) Find the value L_0 such that for $L \leq L_0$, the error probability $P_e \leq 10^{-6}$. *[5pts]*

(c) Find the SNR (function of L). *[5pts]*

(d) Find the minimum SNR that ensures error-free transmission. *[5pts]*

Problem 2 (QAM transmission(15 pts))

QAM transmission is used on an AWGN channel with $\frac{N_0}{2} = 0.01$. The transmitted signal constellation points $\mathbf{s}_i$ for the QAM signal are shown in Figure 3 and are given by:

$$\begin{bmatrix} \pm\frac{\sqrt{3}}{2} \\ \pm\frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \pm 1 \end{bmatrix},$$

with each constellation point equally likely.

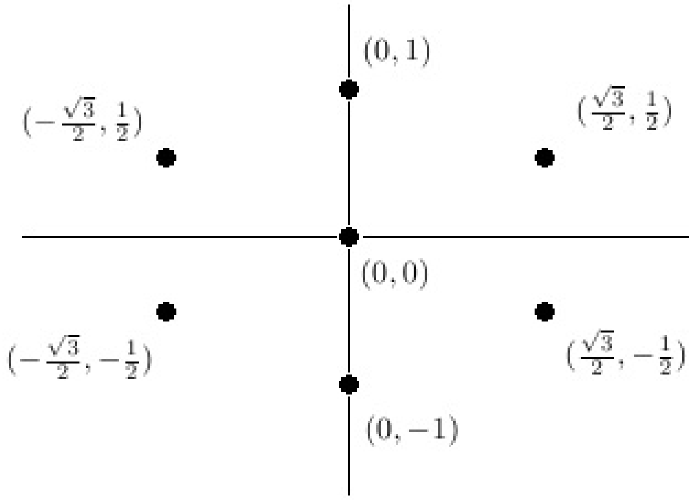


Figure 3: Signal constellation points

(a) Find M (message-set size, *i.e.,* the number of possible messages) and the average energy $E_x = \sum_i p_H(i)||\mathbf{s}_i||^2$ for this constellation. *[4pts]*

(b) For the constellation (see Figure 3) draw the decision regions indicated for an ML detector. *[4pts]*

(c) Find the minimum distance d_{min} for this constellation. *[3pts]*

(d) Find the Nearest Neighbor Union Bound (NNUB) on P_e for the ML detector on this signal constellation. For the NNUB you need to bound $P_e \leq \max_i(N_i)Q(\frac{d_{min}}{2\sigma})$, where N_i is the number of nearest neighbors of constellation point $\mathbf{s}_i$ and $\sigma^2 = \frac{N_0}{2}$. *[4pts]*

Problem 3 (Autocorrelation and PSD(**20 pts**))

Consider a zero-mean wide-sense stationary process $X(t)$ with autocorrelation function $R_X(\tau) = \mathbb{E}[X(t)X(t+\tau)] = e^{-2|\tau|}$. Let $N(t)$ be another zero-mean wide-sense stationary process that is independent of $X(t)$ with autocorrelation $R_N(\tau) = \mathbb{E}[N(t)N(t+\tau)] = e^{-3|\tau|}$.

(a) Find the power spectral density $S_X(f)$ of process $X(t)$. *[7pts]*

(b) Find the power spectral density $S_N(f)$ of process $N(t)$. *[3pts]*

(c) Let $Y(t) = X(t) + N(t)$. Find the autocorrelation function of $Y(t)$. *[6pts]*

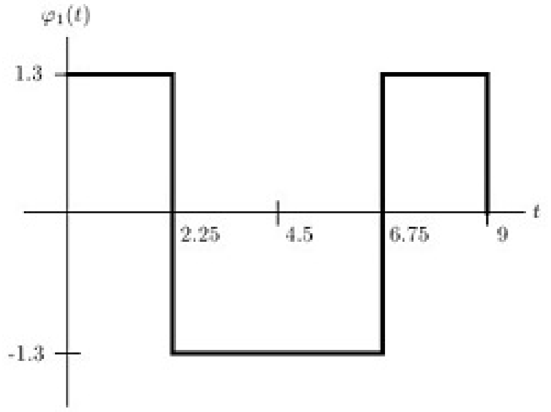
(d) What is the power spectral density $S_Y(f)$ of $Y(t)$? *[4pts]*

Hint: You may find the integrals given in the instructions useful.

Problem 4 (Waveform representations(**25 pts**))

Consider the following two orthogonal basis functions given in Figure 4.

(a) Use the basis functions given Figure 4 to find the modulated waveforms $s_0(t)$ and $s_1(t)$ given the data symbols $\mathbf{s}_0 = [1,\ 1]$ and $\mathbf{s}_1 = [2,\ 1]$. Note that the basis functions in Figure 4 are orthogonal but not orthonormal. It is sufficient to draw $s_0(t)$ and $s_1(t)$. *[7pts]*



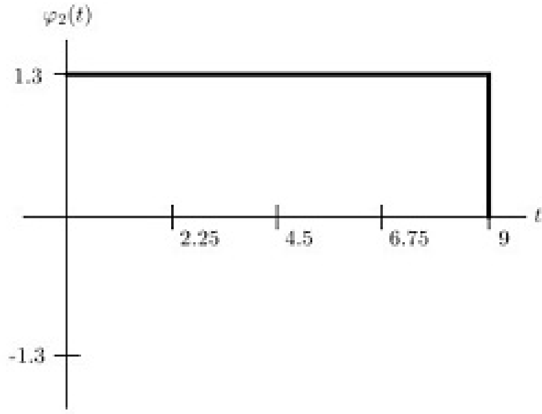
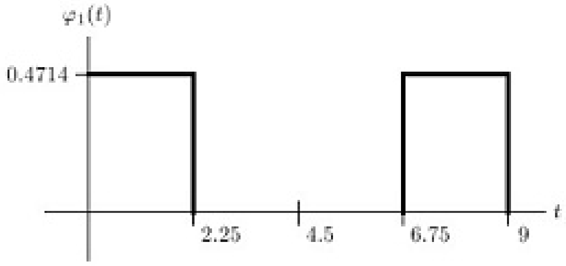


Figure 4: First basis



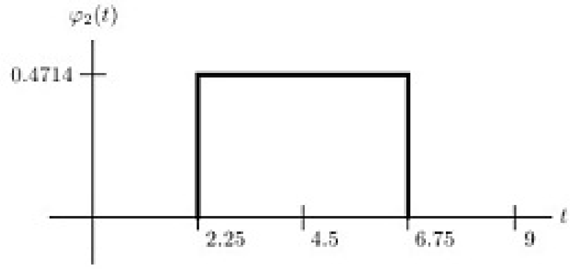


Figure 5: Second basis

(b) For the same $s_0(t)$ and $s_1(t)$, a different set of two orthonormal basis functions is employed, given in Figure 5. Find $\tilde{\mathbf{s}}_0$ which is the representation of $s_0(t)$ in the new basis functions. Find the $\tilde{\mathbf{s}}_1$ which produces $s_1(t)$ in the new basis. Note that though the vector representations change from $\mathbf{s}_i$ to $\tilde{\mathbf{s}}_i$ due to the change in basis functions, the waveforms $s_i(t)$ produced by both representations are the same. *[9pts]*

(c) Now suppose we send the signal over a AWGN channel, *i.e.,* *[9pts]*

$$H = i : y(t) = s_i(t) + N(t),$$

where $N(t)$ is AWGN noise with power spectral density of 10dBm/Hz. Note that 0 dBm corresponds to 1 mW, *i.e.,* x dBm corresponds to power $10^{(x/10)}/1000$ in Watts. Further suppose that both messages are equally likely, *i.e.,* $p_H(0) = \frac{1}{2} = p_H(1)$. Find the error probability for this communication scheme.
Hint: Use the Q-function tables given in the Instructions.

Midterm Solution

Problem 1 (Detection in uniform noise (20 pts))

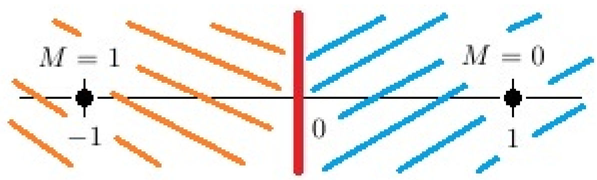
(a) The hypothesis test can be simplified as follows:

$$f_{H|Y}(0|y) \underset{1}{\overset{0}{\gtrless}} f_{H|Y}(1|y)$$
$$p_H(0)f_{Y|H}(y|0) \underset{1}{\overset{0}{\gtrless}} p_H(1)f_{Y|H}(y|1)$$
$$\frac{1}{2}f_Z(y-1) \underset{1}{\overset{0}{\gtrless}} \frac{1}{2}f_Z(y+1)$$
$$f_Z(y-1) \underset{1}{\overset{0}{\gtrless}} f_Z(y+1)$$

From the above, it can be seen that the decision can be made according to:

$$\hat{M} = \begin{cases} 0 & \text{if } y \geq 0 \\ 1 & \text{if } y < 0 \end{cases}$$

with the equivalent decision region given by:



where everything to the right of 0 is decoded as $M = 0$ and everything to the left is decoded as $M = 1$.

(b) The error is the area under the noise pdf that it outside of the correct decision region:

$$P_e = p_H(0)P_e(0) + p_H(1)P_e(1)$$
$$P_e = \frac{1}{2}P_e(0) + \frac{1}{2}P_e(1)$$
$$P_e = (\frac{1}{2} + \frac{1}{2})P_e(0) \quad \text{since } P_e(0) = P_e(1)$$
$$P_e = P_e(0) = \int_1^{\frac{L}{2}} p_Z(z)dz$$
$$P_e = \begin{cases} \int_1^{\frac{L}{2}} \frac{1}{L}dz & \text{if } \frac{L}{2} \geq 1 \\ 0 & \text{otherwise} \end{cases}$$
$$P_e = \begin{cases} \frac{1}{L}(\frac{L}{2} - 1) & \text{if } \frac{L}{2} \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Since the error P_e must be $\leq 10^{-6}$,

$$\frac{1}{L}(\frac{L}{2} - 1) \leq 10^{-6}$$

$$L \leq \frac{1}{\frac{1}{2} - 10^{-6}}$$

$$L_0 = \frac{1}{\frac{1}{2} - 10^{-6}}$$

(c) To find SNR $= \frac{E}{\sigma_Z^2}$, we must find the energy of the signal, E, and the variance of the noise, σ_Z^2.

$$\begin{aligned} E &= p_H(0)\|s_0\|^2 + p_H(1)\|s_1\|^2 \\ &= \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 \\ &= 1 \end{aligned}$$

The power of the noise in the SNR is the variance of Z, so we use the formula for variance to find σ_Z^2:

$$\begin{aligned} \sigma_Z^2 &= E[Z^2] + E[Z]^2 = E[Z^2] \\ &= \int_{-\frac{L}{2}}^{\frac{L}{2}} z^2 p_Z(z) dz \\ &= \int_{-\frac{L}{2}}^{\frac{L}{2}} z^2 \frac{1}{L} dz \\ &= \frac{1}{L} \frac{z^3}{3} \Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{L^2}{12} \end{aligned}$$

Therefore, SNR $= \frac{E}{\sigma_Z^2} = \frac{12}{L^2}$.

(d) The transmission will be error free if $\frac{L}{2} \leq 1$, since at that point, the noise will not overlap the decision region. Then, $L \leq 2$, so SNR $\geq \frac{12}{2^2}$, or SNR ≥ 3. Since $\log_{10}(3)$ is about 0.5, this corresponds to approximately 5 dB.

Problem 2 (QAM TRANSMISSION **(15 pts)**)

(a) The message-set size M is 7, which can be seen from the number of messages in the constellation or by counting the combinations of signals given in vector form.

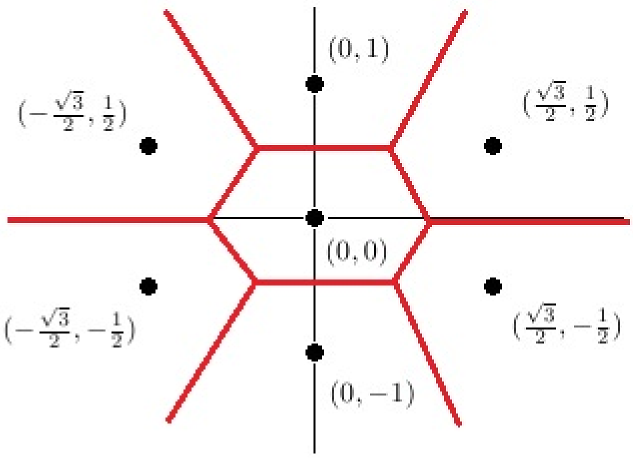
By contrast, some common incorrect answers are given below:
The number of dimensions, usually denoted by N, is 2.
The number of bits required to send the M messages, usually called b or k, is $\log_2(M)$, rounded up, which equals 3.

For the average energy E_x, since all signals are equally-likely, $p_H(i) = \frac{1}{7}\ \forall i$. The energies are given for each signal as follows:

$$\begin{aligned}
&\|[0,0]\|^2 = (0)^2 + (0)^2 = 0 \\
&\|[0,1]\|^2 = (0)^2 + (1)^2 = 1 \\
&\|[0,-1]\|^2 = (0)^2 + (-1)^2 = 1 \\
&\|[\frac{\sqrt{3}}{2}, \frac{1}{2}]\|^2 = (\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 = 1 \\
&\|[-\frac{\sqrt{3}}{2}, \frac{1}{2}]\|^2 = (-\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 = 1 \\
&\|[\frac{\sqrt{3}}{2}, -\frac{1}{2}]\|^2 = (\frac{\sqrt{3}}{2})^2 + (-\frac{1}{2})^2 = 1 \\
&\|[-\frac{\sqrt{3}}{2}, -\frac{1}{2}]\|^2 = (-\frac{\sqrt{3}}{2})^2 + (-\frac{1}{2})^2 = 1
\end{aligned}$$

So the sum of the energies is 6 and the average energy $E_x = \frac{6}{7}$.

(b) The constellation is approximately given by:



To draw the constellation, draw a straight line half-way between each pair of points in the constellation. The lines are half-way between because each point is equally likely. The decision boundaries are lines that form the boundaries between the two points, cut off at their intersections. The boundaries should form a perfect hexagon, with lines radiating out from the corners. Since the figure is not drawn perfectly to scale, it is not exact.

(c) By symmetry, there are four possible distances in the constellation that could be the minimum:

From $(0,0)$ to $(0,1)$ is $\sqrt{(0-0)^2+(1-0)^2}=1$.
From $(0,0)$ to $(\frac{\sqrt{3}}{2},\frac{1}{2})$ is $\sqrt{(\frac{\sqrt{3}}{2}-0)^2+(\frac{1}{2}-0)^2}=1$.
From $(0,1)$ to $(\frac{\sqrt{3}}{2},\frac{1}{2})$ is $\sqrt{(\frac{\sqrt{3}}{2}-0)^2+(\frac{1}{2}-1)^2}=1$.
From $(\frac{\sqrt{3}}{2},\frac{1}{2})$ to $(\frac{\sqrt{3}}{2},-\frac{1}{2})$ is $\sqrt{(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2})^2+(-\frac{1}{2}-\frac{1}{2})^2}=1$.

Since all distances in the figure are equal to 1, $d_{min}=1$.

(d) The point with the maximum number of nearest neighbors is $(0,0)$, which has 6, so $\max_i(N_i)=6$. The problem did not ask for the average number of nearest neighbors. From the problem statement, $\sigma=\sqrt{0.01}=0.1$. Then,

$$P_e \le 6Q(\frac{1}{2*0.1}) = 6Q(5) = 6(2.87*10^{-7}) = 1.72*10^{-6}$$

Problem 3 (Autocorrelation and PSD **(20 pts)**)

(a) The power spectral density of X, $S_X(f)$, is the Fourier transform of the autocorrelation, $R_X(\tau)$. The function $R_X(\tau)=e^{-2|\tau|}$ breaks down into $e^{-2|\tau|}=e^{-2\tau}u(\tau)+e^{2\tau}u(-\tau)$. The Fourier transform is thus:

$$\begin{aligned}S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty} e^{-2|\tau|}e^{-j2\pi f\tau}d\tau \\ &= \int_{0}^{\infty} e^{-2\tau}e^{-j2\pi f\tau}d\tau + \int_{-\infty}^{0} e^{2\tau}e^{-j2\pi f\tau}d\tau \\ &= \frac{1}{2+j2\pi f}+\frac{1}{2-j2\pi f} \\ &= \frac{4}{4+4\pi^2 f^2} = \frac{1}{1+\pi^2 f^2}\end{aligned}$$

where the solutions to the integrals were given in the instructions as:

$\int_0^\infty e^{-at}e^{-j2\pi ft}dt = \frac{1}{a+j2\pi f}$
$\int_{-\infty}^0 e^{at}e^{-j2\pi ft}dt = \frac{1}{a-j2\pi f}$

It is important to simplify the expression into the final form and not leave it in the form containing imaginary numbers. The Fourier transform of a real, even function is also always a real, even function. Therefore, since the autocorrelation function was real and is always even ($R_x(\tau)=R_x(-\tau)$), the PSD should also be real and even, as seen from the final expression.

(b) By the same method as section (a), we obtain

$$S_N(f) = \frac{6}{9+4\pi^2 f^2}$$

(c) For the autocorrelation of $Y(t)$, we simplify the following:

$$
\begin{aligned}
R_Y(\tau) &= E[Y(t)Y(t+\tau)] \\
&= E[(X(t)+N(t))(X(t+\tau)+N(t+\tau))] \\
&= E[X(t)X(t+\tau)+X(t)N(t+\tau)+N(t)X(t+\tau)+N(t)N(t+\tau)] \\
&= E[X(t)X(t+\tau)]+E[X(t)N(t+\tau)]+E[N(t)X(t+\tau)]+E[N(t)N(t+\tau)] \\
&= E[X(t)X(t+\tau)]+E[X(t)]E[N(t+\tau)]+E[N(t)]E[X(t+\tau)]+E[N(t)N(t+\tau)] \\
&\quad \text{(by independence of X and N)} \\
&= E[X(t)X(t+\tau)]+E[N(t)N(t+\tau)] \\
&\quad \text{(because X and N are zero-mean)} \\
&= R_X(\tau)+R_N(\tau) = e^{-2|\tau|}+e^{-3|\tau|}
\end{aligned}
$$

The independence and zero-mean properties should be specified. Independence makes the expectation of the products equal to the product of the expectations, ie. $E[A \cdot B] = E[A]E[B]$. Because X and N also have zero mean, this means $E[X(t)] = E[N(t)] = 0$, which cancels out the cross-terms in the multiplication, leaving just the autocorrelation of X plus the autocorrelation of N.

(d) Again, the power spectral density is the Fourier transform of the autocorrelation. By linearity of the Fourier transform,

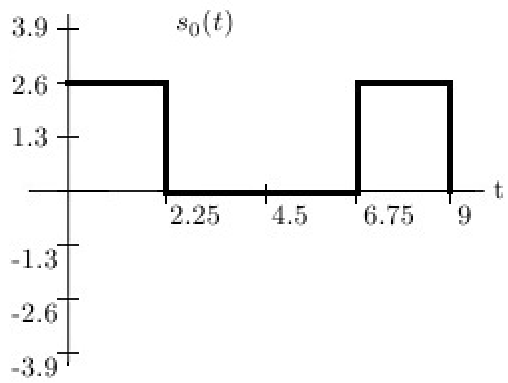
$$
\begin{aligned}
S_Y(f) &= \mathcal{F}\{R_Y(\tau)\} \\
&= \mathcal{F}\{R_X(\tau)+R_N(\tau)\} \\
&= \mathcal{F}\{R_X(\tau)\}+\mathcal{F}\{R_N(\tau)\} \\
&= S_X(f)+S_N(f) \\
&= \frac{4}{4+4\pi^2 f^2}+\frac{6}{9+4\pi^2 f^2}
\end{aligned}
$$

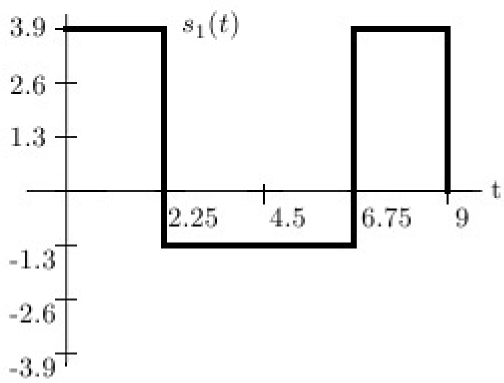
Problem 4 (Waveform representations **(25 pts)**)

(a) The basis functions $\varphi_1(t)$ and $\varphi_2(t)$ were orthogonal, not orthonormal, but the modulated waveforms could be found by

$$
\begin{aligned}
s_0(t) &= \varphi_1(t)+\varphi_2(t) \\
s_1(t) &= 2\varphi_1(t)+\varphi_2(t)
\end{aligned}
$$

which give the following graphs:





(b) The goal is to obtain the waveforms $s_0(t)$ and $s_1(t)$ in the previous section, but with a new basis, such that $s_1(t) = \tilde{s}_{0,0}\varphi_1(t) + \tilde{s}_{0,1}\varphi_2(t)$ and $s_2(t) = \tilde{s}_{1,0}\varphi_1(t) + \tilde{s}_{1,1}\varphi_2(t)$. The answer is

$$\tilde{s}_0 = [\frac{2.6}{0.4714}, 0]$$

$$\tilde{s}_1 = [\frac{3.9}{0.4714}, \frac{-1.3}{0.4714}]$$

(c) To solve this, realize the first basis was not normalized, and thus you must use the vectors found in part (b), which *was* normalized to find the distance between points. In this case, the number in the Q-function could not be easily-calculated without a computer, and the answer must be left unsimplified.

The distance between points is $d = \sqrt{(\frac{3.9}{0.4714} - \frac{2.6}{0.4714})^2 + (\frac{-1.3}{0.4714} - 0)^2} = 3.9$.
Solving for the noise power in Watts: 10 dBm $\Leftrightarrow 10^1/1000$ W $= 10^{-2}$ W $= \sigma^2$.
Then $\sigma = \sqrt{10^{-2}} = 10^{-1}$.
$P_e = Q(\frac{d}{2\sigma}) = Q(\frac{3.9}{2 \cdot 0.1}) = Q(19.5) = 5.91 \cdot 10^{-85}$.
Without a calculator, the distance could be left as $d = \sqrt{2} \cdot \frac{1.3}{0.4714}$, which results in the expression $Q(\sqrt{2} \cdot \frac{1.3}{0.4714} \cdot \frac{1}{0.2})$ or similar.