Communication Systems EE 132A
Prof. Suhas Diggavi

UCLA Winter quarter 2013/2014
Handout #47

FINAL

Thursday, 20th March, 2014, 11:30-14:30
This exam has 6 problems worth 100 points in total plus 5 bonus points.

NAME: ____

UID: ______________________________

PAGES: _______________

Problem	**Max.**	**Awarded**
1	24	
2	10	
3	14+5	
4	19	
5	17	
6	16	
TOTAL	100+5	

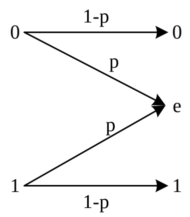
Instructions

- You are allowed to use **2 sheets** of paper for reference. No mobile phones or calculators are allowed in the exam.
- You can attempt the problems in any order as long as it is clear as to which problem is being attempted and which solution to the problem you want us to grade.
- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.
- Write on this sheet in the space provided, the **total number of pages** submitted. The pages in this booklet are numbered already; number the extra pages that you attach. Strike off unused pages.
- The number of pages (including this page) in this booklet is 23. Make sure that your booklet has all the pages printed clearly before starting the exam.

GOOD LUCK! GO BRUINS!

Some **relations/definitions**:

- P_X denotes the probability mass function of a discrete random variable X whose realizations are denoted x. Bold symbols denote the vector analogs.
- $\forall$ denotes "for all"; $\in$ denotes "element(s) of" (set membership); $\cup$ denotes "union"; $\cap$ denotes "intersection".
- In an n-dimensional space, the n unit vectors are represented as $\mathbf{e}_i, i = 1, 2, \ldots, n$ which means for $j = 1, 2, \ldots, n$,
$$e_{i,j} = \begin{cases} 1 & i = j, \\ 0 & \text{else.} \end{cases}$$
- The energy of a vector representation $\boldsymbol{x}$ of a signal $x(t)$ can be calculated as $E = \boldsymbol{x}^T\boldsymbol{x} = \|\boldsymbol{x}\|^2$ where $\|\cdot\|$ denotes the norm or length of a vector.
- The Euclidean distance between two constellation points $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ is $d = \|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.
- The average energy $\mathcal{E}$ of a constellation of vectors $\boldsymbol{s}_i, i = 0, 1, \ldots, M-1$ is given as $\mathcal{E} = \sum_{i=0}^{M-1} P_{\boldsymbol{s}}(\boldsymbol{s}_i)\|\boldsymbol{s}_i\|^2$.
- $Q(a) \triangleq \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\,\mathrm{d}x$, the tail probability of a standard normal distribution which is $\sim \eta(0,1)$. $\sim$ denotes "distributed as".
- The binary erasure channel (BEC) is an abstract channel model which is described as follows. The input is either received correctly or it becomes invalid as an "erasure".



Some **numerical, logarithmic, Q-function, trigonometric** values:

$2 \times 5.375 = 10.75$	$Q(0) = 0.5$	$\sin\left(\frac{\pi}{2}\right) = 1$	$\cos\left(\frac{\pi}{2}\right) = 0$
	$Q(\infty) = 0$	$\sin^2\left(\frac{\pi}{4}\right) = 0.5$	
$\log_{10}\left(\frac{1}{3}\right) = -0.4771$	$Q(-\infty) = 1$	$\sin\left(\frac{\pi}{3}\right) = 0.8660$	$\cos\left(\frac{\pi}{3}\right) = 0.5$
$\log_{10}\left(\frac{1}{2}\right) = -0.3010$	$Q(x) \approx 0, x > 6$	$\sin\left(\frac{2\pi}{3}\right) = 0.8660$	$\cos\left(\frac{2\pi}{3}\right) = -0.5$
$\log_{10}\left(\frac{2}{3}\right) = -0.1761$	$Q(x) \approx 0.5, x \approx 0$	$\sin(\pi + \theta) = -\sin\theta$	$\cos(\pi + \theta) = -\cos\theta$
$\frac{\log\left(\frac{3}{2}\right)}{\log(3)} = 0.3691$	$Q(\sqrt{15}) = 5.375 \times 10^{-5}$	$\sin\left(\frac{\pi}{8}\right) = 0.3827$	$\cos\left(\frac{\pi}{8}\right) = 0.9239$
$\log_{10}(107.5) = 2.0314$	$Q(\sqrt{7.5}) = 0.0031$	$\sin\left(\frac{2\pi}{8}\right) = 0.7071$	$\cos\left(\frac{2\pi}{8}\right) = 0.7071$
$\log_{10}(6200) = 3.7924$	$Q(x) = 1 - Q(-x), \forall x$	$\sin\left(\frac{3\pi}{8}\right) = 0.9239$	$\cos\left(\frac{3\pi}{8}\right) = 0.3827$

Some **identities**:

$\log(a \cdot b) = \log a + \log b$	$2\sin\theta_1\cos\theta_2 = \sin(\theta_1 + \theta_2) + \sin(\theta_1 - \theta_2)$	$e^{j\theta} = \cos\theta + j\sin\theta$
$\log(a/b) = \log a - \log b$	$2\cos\theta_1\cos\theta_2 = \cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2)$	
$k \times \log(a \cdot b) = \log(a \cdot b)^k$	$2\sin\theta_1\sin\theta_2 = \cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)$	
$\log\left(\frac{1}{a}\right) = -\log a$	$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$	
	$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$	

Problem 1 (Overview (24 pts))

In the following, clearly indicate, in your answer, your final choice along with the accompanying justifications/mathematical treatment for a [YES/NO] type question. Also for all the questions, give a clear and supporting mathematical treatment accompanying your answer.

(a) *(Noise-less and noisy observations)* Consider a system which transmits M messages $H_i, i = 0, 1, \ldots, M-1$ as unit vectors along M dimensions i.e. the signaling set, in vector representation, is *[4pts]*

$$\mathbf{s}_i = \mathbf{e}_{i+1}, i = 0, 1, \ldots, M-1.$$

Assume that M is even and denote the transmitted message as

$$\mathbf{X} = \mathbf{s}_i, \text{ if } H = i.$$

A vector noisy channel affects each component of the transmitted vector $\mathbf{X}$ independently; with additive Gaussian noise $N \sim \eta(0, \sigma^2)$ only on the odd numbered dimensions of the vector. The even numbered dimensions are received unaffected. In other words, if $\mathbf{Y}$ is the received message vector, then

$$Y_j = \begin{cases} X_j + N_j & j = 1, 3, 5, \ldots, M-1. \\ X_j & j = 2, 4, 6, \ldots, M. \end{cases} \tag{1}$$

Assuming that the messages at the transmitter are equally likely, design the optimal decoder $d(\mathbf{Y})$ which minimizes the probability of error for this system.

(b) *(Waveform channel design)* Consider a set of 2 "antipodal" waveforms $s_1(t) = -s_0(t)$ used in a binary signaling scheme. Assuming that the channel adds AWGN noise $n(t)$, the optimal waveform receiver can be implemented using just one correlator and a threshold comparator. [YES/NO] Sketch the optimum receiver structure for your answer if $y(t) = s_i(t) + n(t), i = 0$ or 1 denotes the received signal. *Note:* A complete mathematical derivation is not necessary. *[5pts]*

(c) *(Optimum receiver)* Suppose one of M equiprobable signals $x_i(t), i = 0, 1, \ldots, M-1$ is transmitted during a period of time T over an AWGN channel. Each signal is identical to all others in a sub-interval $[t_1, t_2]$, where $0 < t_1 < t_2 < T$. The optimum receiver can ignore the sub-interval $[t_1, t_2]$. [YES/NO] Support your answer with proper justification. *[4pts]*

(d) *(Linear block code over BEC)* Consider the following parity-check matrix of a linear block code: [*5pts*]

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

The decoder for the code, on account of having been deployed to decode over a Binary erasure channel (BEC), received the following word:

$$\mathbf{y} = [e \; e \; 0 \; e \; 0 \; 1 \; 0] \tag{3}$$

where e denotes an erasure. The decoder can decode this received word to a valid codeword. [YES/NO] If yes, what is the decoded codeword? If no, demonstrate why not.

(e) *(OFDM - Time-varying channel)* Consider an OFDM system with $N_c = 3$ sub-carriers implemented for the dispersive channel (ISI channel) given by the following z-transform: *[6pts]*

$$H(z) = 1.33 + 0.33z^{-1} + 0.33z^{-2}. \tag{4}$$

Assume that there is AWGN noise added at the receiver. The transmitter employs a certain transmission strategy and uses all the sub-carriers available to it. Denote the probability of symbol error with this transmit scheme as P_{e}. After some time, the channel condition changes as follows:

$$H_{\mathrm{new}}(z) = 1.67 + 0.17z^{-1} + 0.17z^{-2}. \tag{5}$$

The transmitter continues to employ the same transmit strategy. Assuming that the receiver knows the new channel coefficients, the probability of symbol error for the present channel satisfies $P_{\mathrm{new}} < P_{\mathrm{e}}$. [YES/NO] Support your answer with proper justification.

Problem 2 (Tetrahedral Die (10 pts))

Two four-sided dice, each labeled with $0, 1, 2, 3$ are in a box. For such a die, once we roll it, we agree to observe the number on the bottom face!

- One die is fair (equiprobable outcomes on a roll). Denote this die f.
- The other die is loaded so that 0 is observed with probability $1/2$, the remaining numbers being equally likely. Denote this die u.

In N rolls of a die d which can be either f or u (the die is either fair or unfair), the probability of observing k_i occurrences of face i, where $k_0 + k_1 + k_2 + k_3 = N$ is given by the multinomial distribution:

$$P_{|d}(k_0, k_1, k_2, k_3 \mid d) = \left[\frac{N!}{k_0!k_1!k_2!k_3!}\right] P_{0|d}^{k_0} P_{1|d}^{k_1} P_{2|d}^{k_2} P_{3|d}^{k_3} \tag{6}$$

where $P_{i|d}, i = 0, 1, 2, 3$ are the probabilities of observing face i, conditioned upon die being $d = u$ or $d = f$.

Objective: The experiment consists of picking a die with equal probability, tossing it $N = 10$ times and observing the number of appearances of each of $0, 1, 2$ and 3. The task is to decide which die was chosen.

(a) Formulate the maximum likelihood (ML) decision rule and show that it reduces to *[8pts]*

$$k_0 \underset{\text{fair}}{\overset{\text{unfair}}{\gtrless}} t. \tag{7}$$

The expression for t should either be a numerical value or a numerical expression which can further be evaluated. *Note:* Use $\log_{10}$ values provided in the second page if necessary.

(b) Given that 10 rolls produce observations of $k_0 = 5, k_1 = 2, k_2 = 1, k_3 = 2$, what would you conclude? *[2pts]*

Problem 3 (Unknown Priors (14+5 pts))

You receive a signal $Y = X + Z$, where Z is Gaussian $\sim (0, 1)$ and the prior distribution P_X of the transmitted signal X as per your belief is given by

$$P_X(x) = \begin{cases} q & x = 1, \\ 1 - q & x = -1. \end{cases} \tag{8}$$

(a) Derive the maximum a-posteriori probability (MAP) rule for the hypothesis testing problem of decoding $d(Y) = \hat{X}$ assuming $q = \frac{e}{1+e}$ where "e" refers to the standard exponent. Express the MAP decoding rule as [*3pts*]

$$y \underset{\hat{x}=-1}{\overset{\hat{x}=1}{\gtrless}} t. \tag{9}$$

with an explicit value for t.

(b) Unfortunately, it turns out that the prior distribution of X that you have is incorrect (sorry about that!). Instead, X has the distribution [*6pts*]

$$P_X(x) = \begin{cases} p & x = 1, \\ 1 - p & x = -1, \end{cases} \tag{10}$$

where $0 \leq p \leq 1$. You, however, do not know that you've been given the wrong distribution and are using the decision rule from part (a). What is the resulting error probability?

(c) *The worst prior:* So much for wrong prior information; what value of p yields the largest probability of error for the decision rule from part (a)? [*5pts*]

(d) **Bonus:** *(Exact explanation/mathematical argument required; no partial credit)* [*5pts*]

"Minimax" strategy for combating the worst prior: Someone reveals that your value q is incorrect, but you still do not have any knowledge about what the exact prior p is. Assume that whatever value of q you get to choose, the value of p that provides the worst probability of error for that q will be chosen by nature. This looks hopeless really, but there is something that you can still do:

1. Namely, you can minimize the worst that you can do! What should be your choice of q that minimizes this worst-case probability of error of all choices of q that you can assume?
2. Explicitly state this minimum of all worst-case error probabilities.

Justify your answer rigorously.

Problem 4 (M-ary Signaling and Binary Fading Channel(**19 pts**))

Consider the M-PSK constellation shown in Fig. 1. The constellation is used by a system over an AWGN binary fading channel wherein the noise components of $\mathbf{n}$ along the two dimensions are independent of each other and are identically $\sim (0, \sigma^2)$. The transmitted point $\mathbf{X} = \mathbf{x}$ is selected equiprobably from one of the constellation points $s_m = \sqrt{E}e^{j\frac{2\pi m}{M}}, m = 0, 1, \ldots, M-1$ and the received point is written as

$$\mathbf{y} = \alpha\mathbf{x} + \mathbf{n}. \tag{11}$$

Throughout this problem, assume that the receiver knows α for every received $\mathbf{y}$ and hence it can decode using a modified AWGN MAP (ML) rule; α is a real-valued random variable.

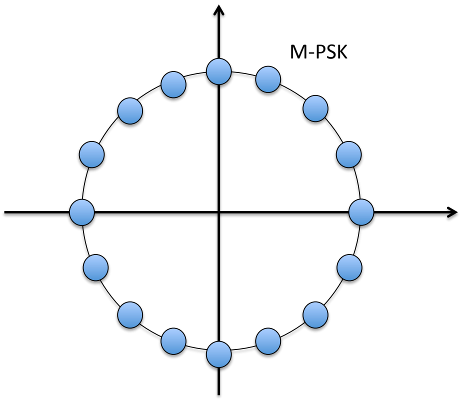


Figure 1: M-PSK

Note: You may find definitions (as a reference) and numerical values on the second page useful while answering the following.

(a) Determine the minimum distance d_{min} of the received constellation ($\alpha\mathbf{X}$) as a function of its average energy and M. For this, first determine the average energy of the received constellation by conditioning on $\alpha = a$ where a is any constant. *[3pts]*

(b) Bound the average probability of symbol error $P_{\mathrm{e}|\alpha}$ using the individual nearest neighbour union bound (INNUB) and express it as a function of the average energy (which is a function of conditioning on the fading parameter $\alpha = a$), M and σ^2. Use the relationship derived in (a) to express d_{min} in terms of the average energy and M. *[4pts]*

(c) Denote $\frac{E}{2\sigma^2} = \mathrm{SNR}$. The binary fading random variable α is distributed as follows: *[5pts]*

$$P_{\alpha^2}(\alpha^2 = a^2) = \begin{cases} \frac{1}{\mathrm{SNR}} & a^2 = 7.5 \times \frac{1}{\mathrm{SNR}}, \\ 1 - \frac{1}{\mathrm{SNR}} & a^2 = 10^3 \times \frac{1}{\mathrm{SNR}}. \end{cases} \tag{12}$$

What is the probability of symbol error P_{e}? *Note:* The probability of symbol error P_{e} is written using the law of total probability as

$$P_{\mathrm{e}} = \sum_{\alpha} P_{\mathrm{e}|\alpha} P(\alpha).$$

(d) Given a symbol error rate requirement of $P_{\mathrm{e}} \leq 10^{-6}$, which constellation among $M = 4$ and $M = 2$ is better and (approximately) by how many dBs? *Hint:* Compare the minimum required operating SNR for achieving the given symbol error rate requirement using the bound obtained. *Note:* A quantity "x" in decibels (dB) is $10 \cdot \log_{10} x$. *[7pts]*

Problem 5 (CONVOLUTIONAL CODE AND OFDM(**17 pts**))

Consider a convolutional code described by the state diagram of Fig. 2. An encoder structure is shown in Fig. 3. The state diagram is complete but the structure of the encoder is incomplete.

→State labels in the state diagram are given in the order b_{j-1}, b_{j-2} i.e. for example, state label "10" corresponds to $b_{j-1} = 1, b_{j-2} = 0$.

→Outputs in the state diagram are given in the order $c_{2j-1}c_{2j}$ i.e. for example, if the encoder is in state "01" and the input is $b_j = 1$, then the output is $c_{2j-1} = 1, c_{2j} = 0$ which is represented as "10".

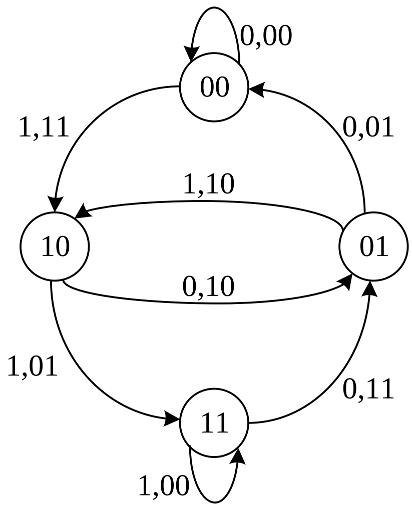


Figure 2: Complete state diagram

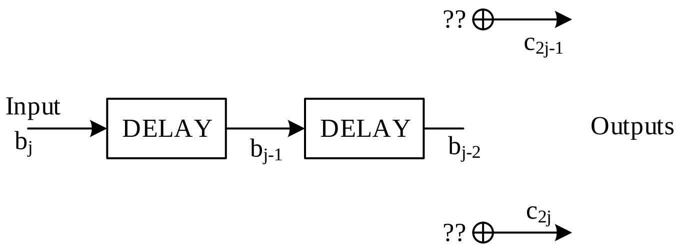


Figure 3: Incomplete encoder structure

(a) Complete the design of this encoder by finding the inputs to the XOR adders which output c_{2j} and c_{2j-1} i.e. copy the incomplete encoder structure and then add to it the necessary connections to the adders such that the outputs shown in the state diagram are generated. [*7pts*]

(b) A four-state rate-1/2 convolutional code (like the one above) is used for transmission over an ISI channel by employing OFDM. We map the output of the convolutional code (using BPSK) as: [*10pts*]

$$\hat{x}_n = \begin{cases} -1 & c_n = 1, \\ 1 & c_n = 0. \end{cases} \tag{13}$$

Premise of the problem: The encoder starts from the all-zero state; which means the delay elements are refreshed to 0's before the first input. We encode 2 bits u_1, u_2 to obtain c_1, c_2, c_3, c_4. At the end, we add two redundant bits u_3, u_4 as input to the convolutional encoder to bring it back to all zeros state.

$\rightarrow$Thus, a total of 8 symbols $\hat{x}_n, n = 1, 2, \ldots, 8$ are transmitted (this is our actual coded symbols mapped using BPSK) corresponding to the codeword sequence $c_1, c_2, \ldots, c_8$. The ISI channel h_l is given by its z-transform as follows:

$$H(z) = 1.5 - 0.5z^{-4}. \tag{14}$$

At the receiver, complex AWGN noise w_n with i.i.d real and imaginary components $\sim \eta(0, \sigma^2)$ is added to the received symbols y_n which is of length $N_c + 2L - 2$.

The problem: Assuming that the transmitter, using $N_c = 8$ sub-carriers, employs required cyclic prefix as it transmits using OFDM, derive the "metric" that needs to be considered at the decoder of the convolutional code to retrieve the maximum likelihood sequence $x'_1, x'_2, \ldots, x'_8$ corresponding to the transmitted sequence $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_8$.

Hint: To do this, first assume that you have processed the received OFDM symbols y_n i.e. assume that you have removed cyclic prefix and the last $L-1$ symbols, where L denotes the number of coefficients in the channel impulse response. Perform the required length DFT of the channel impulse response and obtain a sequence of the N_c data symbols $\hat{x}_n$ which are weighted according to the N_c-point DFT of the channel coefficients plus the AWGN noise. This is the parallel AWGN channel representation obtained by using OFDM. Your task now is to find the metric to be implemented.

Note: For this problem, use the following (standard) definition of a DFT; the DFT V_k of an N_c point sequence $v_n, n = 0, 1, \ldots, N_c - 1$ is:

$$V_k = \sum_{n=0}^{N_c-1} v_n e^{-\frac{j2\pi kn}{N_c}}, k = 0, 1, \ldots, N_c - 1.$$

Provide rigorous explanation for every conceptual step that you take.

Problem 6 (Baseband and Passband Signaling **(16 pts)**)

Let the transmitted bandpass (or passband) signal be given by

$$x(t) = a\cos\left(2\pi f_c t + \frac{\pi}{2}\right) - b\sin\left(2\pi f_c t + \frac{\pi}{2}\right) \tag{15}$$

and $a \in \{1,2\}, b \in \{1,2\}$.

(a) Find the baseband equivalent $x_b(t) = x_1(t) + jx_2(t)$ for the transmitted signal where "j" denotes imaginary component. *[6pts]*

(b) Find the vector representation of the baseband signal and draw the corresponding signal constellation. *[3pts]*

(c) If $a = \begin{cases} 1 & \text{w.p } \frac{1}{2} \\ 2 & \text{w.p } \frac{1}{2} \end{cases}$ and $b = \begin{cases} 1 & \text{w.p } \frac{1}{2} \\ 2 & \text{w.p } \frac{1}{2} \end{cases}$, find the average energy of the baseband signal. *[2pts]*

(d) Is this a minimum energy configuration? If not, how will you modify the constellation so that it is of minimum energy? Draw the minimum energy signal constellation if your answer was no. *[5pts]*