# FINAL EXAM

Monday, 19th March, 2012, 15:00-18:00
This exam has 6 problems and 100 points in total.

## Instructions

- You are allowed to use 2 sheets of paper for reference. No mobile phones or calculators are allowed in the exam.
- You can attempt the problems in any order as long as it is clear which problem is being attempted and which solution to the problem you want us to grade.
- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.
- Please solve every problem on **separate paper sheets**.
- It is your responsibility to **number the pages** of your solutions and write on the first sheet the **total number of pages** submitted.

Q-function Values:

| | |
|---|---|
| $Q(0) = 0.5$ | $Q(4) = 3.17 \times 10^{-5}$ |
| $Q(0.5) = 0.31$ | $Q(4.4) = 5 \times 10^{-6}$ |
| $Q(2/3) = 0.25$ | $Q(5) = 2.87 \times 10^{-7}$ |
| $Q(1) = 0.16$ | $Q(7.5) = 3.22 \times 10^{-14}$ |
| $Q(1.5) = 0.067$ | $Q(10) = 7.77 \times 10^{-24}$ |
| $Q(2) = 0.023$ | $Q(12.5) = 3.86 \times 10^{-36}$ |
| $Q(2.5) = 0.0062$ | $Q(15) = 3.85 \times 10^{-51}$ |
| $Q(3) = 0.0013$ | $Q(25) = 3.38 \times 10^{-138}$ |
| $Q(3.5) = 0.00023$ | $Q(\infty) = 0$ |

GOOD LUCK!

## Problem 1 (COMPARING PSK SYSTEMS (17 pts))

Suppose we transmit a $M$-PSK constellation, where for message $m \in \{0, \dots, M-1\}$, we transmit the 2-dimensional signal $s_m = \sqrt{E} e^{j\frac{2\pi m}{M}}$. The transmission is over a AWGN channel and the received (discrete-time) signal is

$$\mathbf{y} = \mathbf{s} + \mathbf{z}$$

where $\mathbf{z}$ is a 2 dimensional zero-mean Gaussian with covariance $\sigma^2\mathbf{I}$.

(a) Find an expression for the probability of symbol error for the 4-PSK system, as a function of average transmit energy $E$ and $\sigma^2$. *[5pts]*

(b) Upper bound the error probability using the Nearest-Neighbor Union Bound (NNUB). *[3pts]*

(c) Using the bound in $(b)$ find the SNR $\frac{E}{\sigma^2}$ for which the symbol error would be $10^{-5}$. *[3pts]*

(d) Now consider a 8-PSK system. Assume that the energy per transmitted symbol is $3E/2$ (this will ensure that bit energy comparisons with the 4-PSK system are fair). Bound the the symbol error rate, using the NNUB for the 8-PSK system in terms of SNR. Compare this with the error rate in $(b)$ for the *same* energy per bit. *[6pts]*

## Problem 2 ( The Z-Channel (16 pts))

Consider the following binary channel shown in Figure 1.

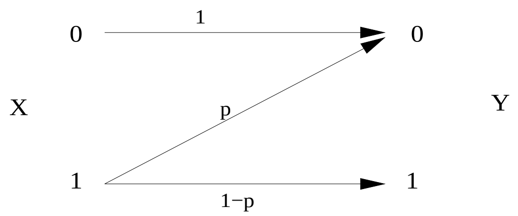



Figure 1: The Z-channel.

(a) Let us consider a single transmission with $\mathbb{P}[X=0]=q$, $\mathbb{P}[X=1]=1-q$. Assume that the prior $q$ is known at the decoder and of course $p$ is known as well. Given observation $Y$, give the decision rule that minimizes the probability of error, i.e., $\mathbb{P}\left[\hat{X}\neq X\right]$. *[10pts]*
You can assume that $q\in[0,\frac{1}{2}]$ and $p\in[0,\frac{1}{2}]$.
**Hint:** Consider cases when $\frac{q}{1-q}\lesseqgtr p$.

(b) Now, consider successive transmissions where the **same** input symbol $X$ is transmitted $n$ times over independent realizations of this channel. As before, consider the priors to be $\mathbb{P}[X=0]=q$, $\mathbb{P}[X=1]=1-q$, again known at the decoder. Given the observations $Y_1,\ldots,Y_n$, give the decision rule that minimizes the error probability. *[6pts]*

## Problem 3 (PAM constellation design (21 pts))

Suppose you can send a PAM constellation depicted in Figure 2, and the problem is in choosing the appropriate constellation according to the requirements. Suppose we are transmitting over an AWGN channel

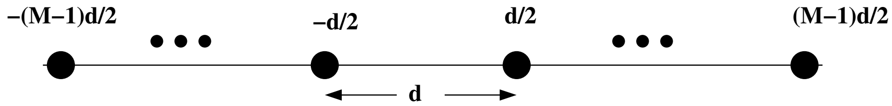



Figure 2: The M-PAM constellation.

$$y[k]=s[k]+z[k],$$

where $z\sim\eta(0,\sigma^2)$. Suppose we operate at SNR of $30dB$. Recall that as done in class, for PAM constellations the signal energy is

$$\mathcal{E}=\frac{d^2}{12}(M^2-1),$$

where $d$ is the minimum distance. The SNR is $\frac{\mathcal{E}}{\sigma^2}$.

(a) If we sent $b$ bits per transmission, *i.e.,* $M = 2^b$, then the minimum distance $d = \sqrt{\frac{12\mathcal{E}}{4^b-1}}$. *[10pts]*
Denote $\mathcal{E}(b)$ as the energy needed to send $b$ bits for a given minimum distance $d$, *i.e.,*

$$d = \sqrt{\frac{12\mathcal{E}(b)}{4^b - 1}} \tag{1}$$

If we want to maintain the same minimum distance, then show that the energy needed for sending $b+1$ bits can be written as,

$$\mathcal{E}(b+1) = 4\mathcal{E}(b) + \frac{d^2}{4}$$

*Hint:* Note that as we need $d$ to remain constant, we can use (1) to obtain this result.

(b) Find the largest constellation with integer $b$ for which $P_e < 5.74 \times 10^{-7}$, *i.e.,* number of messages is $M = 2^b$. *[6pts]*
*Hint:* You can use the Nearest Neighbor union bound for PAM constellations (where the maximum number of Nearest Neighbors is 2) as an approximation for $P_e$.

(c) How much improvement in SNR would you need to increase the number of bits per transmission by 2 from the value that you found in part (*b*) for the same error probability? That is we want to send $b+2$ bits per transmission, where $b$ is the value found in part (*b*). *[5pts]*

## Problem 4 (OFDM (15 pts))

Consider a discrete ISI channel described by the model

$$y[k] = s[k] * h[k] + z[k] = \sum_{n=0}^{L-1} h[n]s[k-n] + z[k]$$

where $z[k]$ is AWGN noise with variance $\frac{N_0}{2}$, $s[k]$ is the transmitted signal point, $y[k]$ is the received signal, and $h[k]$ is the transfer function of the channel, which has two taps and a z-transform given by

$$H(z) = 1 + 0.9z^{-1} \tag{2}$$

(a) Suppose we use OFDM with $N_c = 4$ for this channel. What is the length of the cyclic prefix that is needed for this channel? *[2pts]*

(b) Now, we have learned in class that by using the cyclic prefix of the appropriate length (given in (*a*) above) we can obtain an equivalent set of parallel channels: *[5pts]*

$$\tilde{Y}[m] = \tilde{H}[m]\tilde{S}[m] + \tilde{Z}[m], m = 0, \ldots, N_c - 1,$$

where

$$\tilde{H}[m] = \sum_{l=0}^{L-1} h[l]e^{-j\frac{2\pi ml}{N_c}}.$$

For the channel given in (2), evaluate explicitly $\tilde{H}[m]$, using the same $N_c$ we had specified in (*a*).

(c) Suppose that for each sub-channel $m \in \{0, 1, 2, 3\}$, we are going to send a signal with transmit energy $\mathbb{E}[|\tilde{S}|^2] = \mathcal{E}$, therefore yielding a received SNR of $\frac{|\tilde{H}[m]|^2 \mathcal{E}}{N_0/2}$. Find the received SNRs for the channel in (2). [8pts]
*Hint:* Use your calculation from (*b*) above. Note that for a complex number $c = a + jb$, $|c|^2 = |a|^2 + |b|^2$ and also $e^{j\theta} = \cos\theta + j\sin\theta$.

## Problem 5 (Convolution codes (14 pts))

Consider the convolutional encoder and the state diagram given in Figures 3 and 4 respectively. The state diagram in Figure 3 is complete, but the block diagram in Figure 4 is not complete. State labels in the state diagram are given in the order as in class, *i.e.,* $b_{j-1}b_{j-2}$ (in other words, state 10 in the state diagram corresponds to the state where $b_{j-1} = 1, b_{j-2} = 0$. Outputs in the state diagram are given in the order $c_{2j-1}c_{2j}$. In other words, if the encoder is in state 01, *i.e.,* $b_{j-1} = 0, b_{j-2} = 1$ and the input $b_j = 1$, then the output is 10, *i.e.,* $c_{2j-1} = 1 = b_j$ and $c_{2j} = 0$.

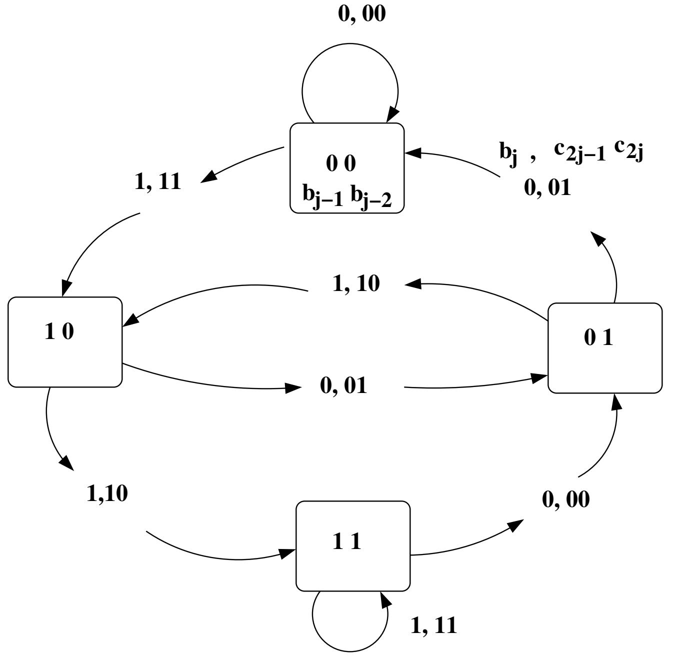



Figure 3: The state diagram of the convolutional code.

(a) Complete the design of this encoder by finding the inputs to the XOR adder which output $c_{2j}$. In other words, for your answer, copy the block diagram given in Figure 4, and then add to it the necessary connections to the adder such that the outputs shown on the diagram are generated. [8pts]

(b) Draw the detour flow graph, and label the edges by the input weight using the symbol $I$, and the output weight using the symbol $D$. [6pts]

## Problem 6 (Fading channel (17 pts))

Suppose we want to send a 4-PAM signal constellation, illustrated in Figure 5 over a slowly fading wireless channel. The equivalent discrete-time channel as seen in the lectures, can be modeled as:

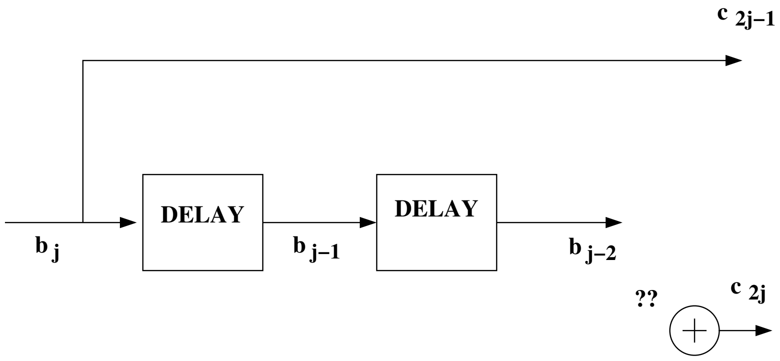



Figure 4: Representation of the convolutional code in terms of the function of the input bits. You need to complete this representation using the state representation in Figure 3.

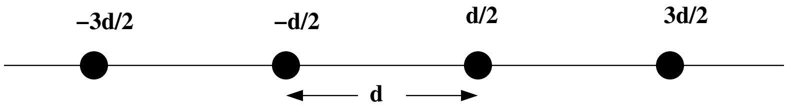



Figure 5: The 4-PAM constellation.

$$y[k] = hs[k] + z[k], \tag{3}$$

where $z[k] = z^I[k] + jz^Q[k]$ is complex additive Gaussian noise, with $z^I[k] \sim \eta(0, \frac{\sigma^2}{2})$, $z^Q[k] \sim \eta(0, \frac{\sigma^2}{2})$, $z^I[k]$ and $z^Q[k]$ being independent of each other.

(a) Suppose the channel $h$ is a *fixed* and known constant. Then find the optimal decision regions to minimize error probability. *[4pts]*

(b) Find the error probability for the decision rule in $(a)$, when the channel $h$ is a fixed constant and known. *[2pts]*

(c) Now suppose we have a statistical model for the wireless channel. The fading channel $h$ is statistically modeled as a complex Gaussian channel, and therefore $|h|^2$ is exponentially distributed, *i.e.,* *[7pts]*

$$f_{|h|^2}(a) = e^{-a}, \;\; a \geq 0. \tag{4}$$

Since the channel is assumed to be known to the receiver, the optimal decision rule remains the same as in $(a)$.

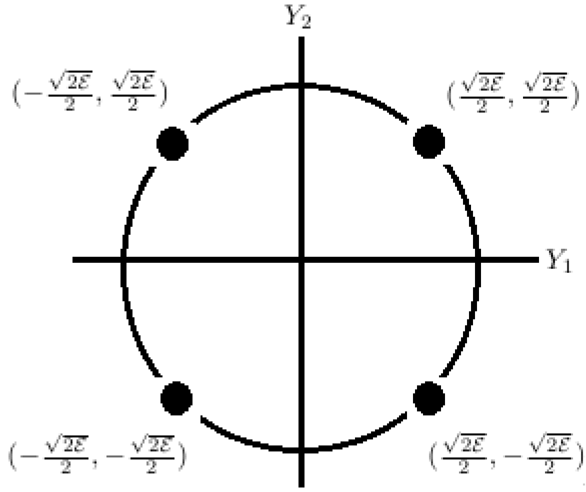
Find the average error probability, *i.e.,* $\mathbb{E}_{|h|^2}[P_e(|h|^2)]$.
*Hint:* For a given $h$ one can write the error probability in terms of a Q-function that depends on $|h|^2$. One can then use the following identity $\mathbb{E}[Q(|h|U)] = \frac{1}{2}\left[1 - \sqrt{\frac{U^2}{U^2+2}}\right]$, for the statistical model described in (4). For example in HW 8 you did this calculation for binary transmission and found that the average error probability was $\mathbb{E}[Q(\sqrt{|h|^2 2\text{SNR}})] = \frac{1}{2}\left[1 - \sqrt{\frac{SNR}{SNR+1}}\right]$.

(d) Use the approximation $\frac{1}{\sqrt{1+x}} \approx 1 - \frac{x}{2}$ for small $x$, to demonstrate the behavior of the error probability calculated in $(d)$ for large SNR. Using this approximation determine the SNR needed for error probability of $10^{-3}$. *[4pts]*

# Final Solution

## Problem 1 (Comparing PSK Systems)

(a) The absolute probability of error can be found by considering the following signal constellation for 4-PSK (the probability of error does not change with rotations around the origin, so any rotation of the constellation is equivalent).




To evaluate the error for this constellation, first find the probability of correct decision $P_c$ and then from this find the probability of error $P_e = 1 - P_c$. Each signal is equally-probable and each has the same energy, so the average error for the whole system will be the same as for any of the individual symbols. We choose the symbol in the top right corner and evaluate $P_c$,

$$\begin{aligned}
P_c &= P(Z_1 > -\frac{\sqrt{2\mathcal{E}}}{2}) \cdot P(Z_2 > -\frac{\sqrt{2\mathcal{E}}}{2}) \\
&= Q(-\frac{\sqrt{2\mathcal{E}}}{2\sigma}) \cdot Q(-\frac{\sqrt{2\mathcal{E}}}{2\sigma}) \\
&= (1 - Q(\frac{\sqrt{2\mathcal{E}}}{2\sigma}))^2 \\
&= 1 - 2Q(\frac{\sqrt{2\mathcal{E}}}{2\sigma}) + Q^2(\frac{\sqrt{2\mathcal{E}}}{2\sigma}) \\
P_e &= 1 - P_c \\
&= 2Q(\frac{\sqrt{2\mathcal{E}}}{2\sigma}) - Q^2(\frac{\sqrt{2\mathcal{E}}}{2\sigma})
\end{aligned}$$

(b) Each signal has 2 nearest neighbors at a distance $d_{min} = \sqrt{2\mathcal{E}}$. Therefore the NNUB is

$$P_e \leq 2Q(\frac{\sqrt{2\mathcal{E}}}{2\sigma})$$

(c) Using $P_e = 10^{-5}$ and the NNUB found in part (b) we find

$$10^{-5} = 2Q(\frac{\sqrt{2\mathcal{E}}}{2\sigma})$$

$$5 \cdot 10^{-6} = Q(\frac{\sqrt{2\mathcal{E}}}{2\sigma})$$

$$5 \cdot 10^{-6} = Q(\frac{\sqrt{2}}{2}\sqrt{\text{SNR}})$$

where SNR is $\mathcal{E}/\sigma^2$. From the Q-table provided, it could be determined that $Q(4.4) = 5 \cdot 10^{-6}$. Therefore

$$\frac{\sqrt{2}}{2}\sqrt{\text{SNR}} = 4.4$$

$$\frac{1}{2} \cdot \text{SNR} = (4.4)^2$$

$$\text{SNR} = 2(4.4)^2 \approx 38.72$$

(d) For the 8-PSK system, the distance between each two adjacent points is $d = 2\sqrt{\mathcal{E}}\sin(\pi/8)$. Here, $\mathcal{E} = 3\mathcal{E}_a/2$, where $\mathcal{E}_a$ is the energy from part (a). There are 2 nearest neighbors in this constellation as well, giving a NNUB of

$$P_e \leq 2Q\left(\sqrt{\frac{3\mathcal{E}_a}{2\sigma^2}}\sin(\pi/8)\right) = 2Q\left(\sqrt{\frac{3}{2}\text{SNR}} \cdot \sin(\pi/8)\right)$$

The distance between points in 8-PSK is closer than for the 4-PSK system, so the error rate is higher for the same energy.

## Problem 2 (The Z-Channel)

(a) Writing out the MAP hypothesis test for this scenario gives

$$P_{X|Y}(0|y) \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} P_{X|Y}(1|y)$$

$$P_X(0)P_{Y|X}(y|0) \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} P_X(1)P_{Y|X}(y|1)$$

$$\frac{P_X(0)}{P_X(1)} \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} \frac{P_{Y|X}(y|1)}{P_{Y|X}(y|0)}$$

Now we must evaluate what each of these four values are. From the problem statement, $P_X(1) = 1 - q$ and $P_X(0) = q$. By evaluating the Z-channel diagram, we can write the other two values as

$$P_{Y|X}(y|0) = \begin{cases} 1 & \text{if } Y = 0 \\ 0 & \text{if } Y = 1 \end{cases}$$

$$P_{Y|X}(y|1) = \begin{cases} p & \text{if } Y = 0 \\ 1-p & \text{if } Y = 1 \end{cases}$$

So if $Y = 0$ (the received value is 0), the hypothesis test becomes

$$\frac{q}{1-q} \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} p$$

and if $Y = 1$, we will always decide $\hat{X} = 1$ because from the diagram, if $Y = 1$, it can only come from $X = 1$. In summary the decision is made according to

$$\hat{X} = \begin{cases} 0 & \text{if } Y = 0,\ \frac{q}{1-q} > p \\ 1 & \text{otherwise} \end{cases}$$

(b) Let the number of 1's received be denoted as $k$, where $Z = k$ is a sufficient statistic for decoding. Now we can write the MAP test for this receiver as

$$P_{X|Z}(0|k) \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} P_{X|Z}(1|k)$$

$$\frac{P_X(0)}{P_X(1)} \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} \frac{P_{Z|X}(k|1)}{P_{Z|X}(k|0)}$$

As before the ratio on the left side is $\frac{q}{1-q}$. If any 1's are received, we must still decide that $X = 1$ because there is no other way to receive $Y = 1$. In the case that $k = 0$, then we decide $\hat{X} = 0$ when

$$\frac{q}{1-q} \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} \frac{P_{Z|X}(0|1)}{P_{Z|X}(0|0)}$$

$$\frac{q}{1-q} \underset{\hat{X}=1}{\overset{\hat{X}=0}{\gtrless}} p^n$$

In summary, the decision is

$$\hat{X} = \begin{cases} 0 & \text{if } k = 0,\ \frac{q}{1-q} > p^n \\ 1 & \text{otherwise} \end{cases}$$

## Problem 3 (PAM Constellation Design)

(a) Since $d$ does not change when increasing the number of bits, as given in the problem, equate the expressions for $d$ for both $\mathcal{E}(b)$ and $\mathcal{E}(b+1)$ and simplify to obtain the desired expression,

$$d = \sqrt{\frac{12\mathcal{E}(b)}{4^b - 1}} = \sqrt{\frac{12\mathcal{E}(b+1)}{4^{b+1} - 1}}$$

Taking the square of both sides and dividing by 12,

$$\frac{\mathcal{E}(b)}{4^b - 1} = \frac{\mathcal{E}(b+1)}{4 \cdot 4^b - 1}$$

Finally, solving for $\mathcal{E}(b+1)$ in terms of $\mathcal{E}(b)$,

$$\begin{aligned}
\mathcal{E}(b+1) &= \mathcal{E}(b)\frac{4 \cdot 4^b - 1}{4^b - 1} \\
&= \mathcal{E}(b)\frac{4(4^b - \frac{1}{4})}{4^b - 1} \\
&= 4\mathcal{E}(b)\frac{(4^b - 1 + \frac{3}{4})}{4^b - 1} \\
&= 4\mathcal{E}(b) + \frac{3\mathcal{E}(b)}{4^b - 1} \\
&= 4\mathcal{E}(b) + \frac{d^2}{4}
\end{aligned}$$

(b) The NNUB for a PAM constellation, assuming that the number of nearest neighbors is 2, is given by

$$P_e \leq 2Q(\frac{d}{2\sigma})$$

where $d$ is given in part (a) of this problem. Set $P_e = 5.74 \cdot 10^{-7}$ and solve:

$$5.74 \cdot 10^{-7} = 2Q\left(\frac{1}{2\sigma}\sqrt{\frac{12\mathcal{E}}{4^b - 1}}\right)$$

$$2.87 \cdot 10^{-7} = Q\left(\sqrt{\frac{3\mathcal{E}}{\sigma^2(4^b - 1)}}\right)$$

$$2.87 \cdot 10^{-7} = Q\left(\sqrt{\text{SNR}}\sqrt{\frac{3}{4^b - 1}}\right)$$

From the Q-function table provided, $Q(5) = 2.87 \cdot 10^{-7}$, so

$$\sqrt{\text{SNR}}\sqrt{\frac{3}{4^b - 1}} = 5$$

$$\frac{3 \cdot \text{SNR}}{4^b - 1} = 25$$

$$b = \log_4(\frac{3 \cdot \text{SNR}}{25} + 1)$$

The SNR was provided in the problem statement as SNR $= 30$ dB $= 1000$. Plugging in, $b = \log_4(121)$, so $b$ is somewhere between 3 and 4. To decide whether to pick 3 or 4, consider that a larger value of $b$ makes the denominator in the $Q$ function larger, making the argument smaller, which makes the error larger. Likewise, picking a smaller value of $b$ makes the error smaller, so $b = 3$ is the maximum-allowable numbers of bits.

(c) There are several ways to solve this section. One would be to use the formula from part (a) and solve recursively. Another way would be to use the derivation from part (b), plugging in $b = 3 + 2 = 5$ and solve for SNR. This gives

$$\frac{3 \cdot \text{SNR}}{4^5 - 1} = 25$$

$$\mathrm{SNR} = \frac{25}{3} \cdot 1023 = 8525 \approx 39 \text{ dB}$$

An approximation could be make without a calculator. The SNR should be increased by 9 dB. A typical approximation is that increasing the number of bits by 1 requires a 6 dB increase in SNR. This holds in this case because the the exact number of bits calculated in part (c) was somewhere between 3 and 4, so a 9 dB increase makes sense.

## Problem 4 (OFDM)

(a) The length of the cyclic prefix is the number of taps from delays in the system, so it is 1.

(b) By entering known values and evaluating the sum, we obtain

$$\begin{aligned}\tilde{H}[m] &= h[0]e^0 + h[1]e^{-j\frac{\pi m}{2}} \\ &= 1 + 0.9e^{-j\frac{\pi m}{2}}\end{aligned}$$

(c) We can find the transfer function for each sub-channel $m \in \{0, 1, 2, 3\}$ as

$\tilde{H}[0] = 1 + 0.9e^0 = 1.9$
$\tilde{H}[1] = 1 + 0.9e^{-j\pi/2} = 1 - 0.9j$
$\tilde{H}[2] = 1 + 0.9e^{-j\pi} = 1 - 0.9 = 0.1$
$\tilde{H}[3] = 1 + 0.9e^{-j3\pi/2} = 1 + 0.9j$

Now evaluate the SNR as $\frac{|\tilde{H}[m]|^2\mathcal{E}}{N_0/2}$ for each sub-channel. First find the magnitude squared of the sub-channels:

$|\tilde{H}[0]|^2 = 1.9^2 = 3.61$
$|\tilde{H}[1]|^2 = 1^2 + 0.9^2 = 1.81$
$|\tilde{H}[2]|^2 = 0.1^2 = 0.01$
$|\tilde{H}[3]|^2 = 1^2 + 0.9^2 = 1.81$

Now evaluate the SNRs for each sub-channel:

$\mathrm{SNR}_0 = 2(3.61)\frac{\mathcal{E}}{N_0} = 7.22\frac{\mathcal{E}}{N_0}$
$\mathrm{SNR}_1 = 2(1.81)\frac{\mathcal{E}}{N_0} = 3.62\frac{\mathcal{E}}{N_0}$
$\mathrm{SNR}_2 = 2(0.01)\frac{\mathcal{E}}{N_0} = 0.02\frac{\mathcal{E}}{N_0}$
$\mathrm{SNR}_3 = 2(1.81)\frac{\mathcal{E}}{N_0} = 3.62\frac{\mathcal{E}}{N_0}$

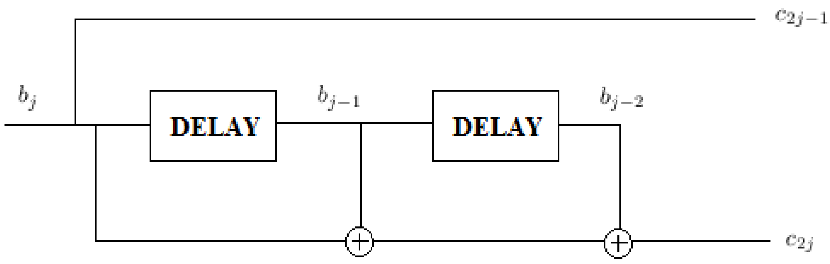
## Problem 5 (Convolution Codes)

(a) From examining the outputs given in the state diagram, it is seen that the second output is $c_{2j} = b_j \oplus b_{j-1} \oplus b_{j-2}$. This can be verified by looking at each of the branches:

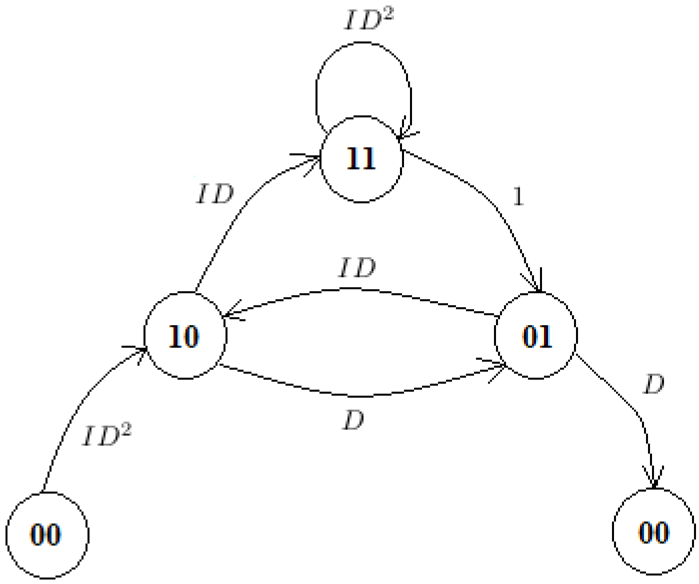
From 00 to 00: $c_{2j} = b_j \oplus b_{j-1} \oplus b_{j-2} = 0 \oplus 0 \oplus 0 = 0$
From 00 to 10: $c_{2j} = 1 \oplus 0 \oplus 0 = 1$
From 10 to 01: $c_{2j} = 0 \oplus 1 \oplus 0 = 1$
From 10 to 11: $c_{2j} = 1 \oplus 1 \oplus 0 = 0$
From 11 to 01: $c_{2j} = 0 \oplus 1 \oplus 1 = 0$

From 11 to 11: $c_{2j} = 1 \oplus 1 \oplus 1 = 1$
From 01 to 00: $c_{2j} = 0 \oplus 0 \oplus 1 = 1$
From 01 to 10: $c_{2j} = 1 \oplus 0 \oplus 1 = 0$

The figure below shows the correct block diagram.

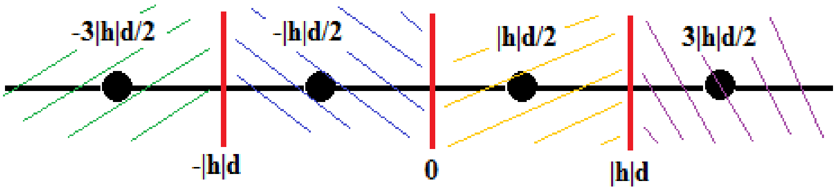



(b) The reference path is the all-zero codeword. Therefore, for no errors the input should always be 0 and the output should always be 00. The detour flowgraph is created by examining all paths through the state diagram the code can take after an initial input error. Each branch is labeled with $I^iD^d$, where $i$ is the number of input bits that have changed for that branch from the reference (or equivalently the number of input 1's), and $d$ is the umber of output bits that have changed for that branch from the reference (number of output 1's). The figure below shows the correct labeling (note that $I^0D^0 = 1$).



## Problem 6 (Fading Channel)

(a) A fixed, constant $h$ just has the effect of scaling the amplitude of the signals by a constant value. An equivalent channel would look like




with the decision regions separated by vertical lines and shaded. Formally, the decision regions are

$$\hat{s}[k] = \begin{cases} 0 & \text{if } y[k] \leq -|h|d \\ 1 & \text{if } -|h|d < y[k] \leq 0 \\ 2 & \text{if } 0 < y[k] \leq |h|d \\ 3 & \text{if } y[k] > |h|d \end{cases}$$

if the points are labeled $s = \{0, 1, 2, 3\}$ from left to right.

(b) The effective noise value of the channel is $\sigma^2_{Eff} = \sigma^2_{Re} + \sigma^2_{Im} = \frac{\sigma^2}{2} + \frac{\sigma^2}{2} = \sigma^2$. Therefore, the error for points 0 and 3 is

$$P_e(0) = P_e(3) = Q\left(\frac{|h|d}{2\sigma}\right)$$

since they only have one adjacent point. For points 1 and 2,

$$P_e(1) = P_e(2) = 2Q\left(\frac{|h|d}{2\sigma}\right)$$

since they have two adjacent points. Averaging based on the assumption that all signals are equally-likely,

$$P_e = \frac{3}{2}Q\left(\frac{|h|d}{2\sigma}\right)$$

(c) Since for binary transmission, the error probability for a fixed $h$ is

$$P_e = Q\left(\frac{|h|d}{2\sigma}\right)$$

which produced an average energy $\mathbb{E}[Q(\frac{|h|d}{2\sigma})]$ of

$$\frac{1}{2}\left[1 - \sqrt{\frac{\text{SNR}}{1 + \text{SNR}}}\right]$$

the 4-PAM error is just the binary case scaled by the constant $3/2$, so

$$\mathbb{E}[\frac{3}{2}Q\left(\frac{|h|d}{2\sigma}\right)] = \frac{3}{2} \cdot \frac{1}{2}\left[1 - \sqrt{\frac{\text{SNR}}{1 + \text{SNR}}}\right] = \frac{3}{4}\left[1 - \sqrt{\frac{\text{SNR}}{1 + \text{SNR}}}\right]$$

(d) Dividing top and bottom of the fraction by SNR and using $x = 1/\text{SNR}$, the approximation $\frac{1}{\sqrt{1+x}} \approx 1 - \frac{x}{2}$ can be applied to make the approximation

$$P_e = \frac{3}{4}\left[1 - \frac{1}{\sqrt{1+x}}\right] \approx \frac{3}{4}[1 - (1 - \frac{x}{2})] = \frac{3}{4} \cdot \frac{x}{2}$$

Substituting back for $x$ and setting $P_e = 10^{-3}$,

$$10^{-3} \approx \frac{3}{8 \cdot \text{SNR}}$$

So $\text{SNR} \approx 3000/8 = 375 \approx 25.7$ dB.