EE 131A **Final Exam 1(25 pts; Closed book; 90 min.)** **Winter 2013**

K. Yao

1. **a.** Consider two r.v.'s X and Y with a joint pdf of $f_{X,Y}(x,y) = bxy,\ 0 \le x \le 1,\ 0 \le y \le 1$.
   (i.) Find b. (1 pt)
   (ii.) Find $f_X(x)$ and $f_Y(y)$. (2 pts)
   (iii.) Are these two r.v.'s independent? (1 pt)

   b. Consider two r.v.'s X and Y with a joint pdf of $f_{X,Y}(x,y) = cxy,\ 0 \le x \le y \le 1$.
   (i.) Find c. (1 pt)
   (ii.) Find $f_X(x)$ and $f_Y(y)$. (2 pts)
   (iii.) Are these two r.v.'s independent? (1 pt)

2. Suppose we have $m = 4$ men and $n = 3$ women. How many different ways can we form a team of $r = 3$ persons (without regard to the number of men or women in the team)?

   a. Express your answer in the form of the combinatorial notation C_x^y. What are the integers y and x expressed in terms of m, n, r? Find the numerical value of C_x^y. (2 pts)

   b. Express your answer in the form of $\sum_k C_b^a C_e^d$. How many terms are there in the sum and what are b, c, d, e and k in terms of m, n, r? Find the numerical value of $\sum_k C_b^a C_e^d$. (4 pts)

3. Let X be a uniform r.v. on $(0, a)$, where $0 < a$, and let B be the event $0 \le X \le b$, where $0 < b < a$.

   a. Find the conditional cdf $F_{X|B}(x|B)$, for $0 < b \le x$ and $0 \le x \le b$. (3 pts)

   b. Find the conditional pdf $f_{X|B}(x|B)$ for $0 \le x \le b$. (2 pts)

4. Consider the discrete r.v. X with a pmf defined by $p_X(X = k) = (1/2)^k, k = 1, 2, \ldots$. Let the discrete r.v. Y defined by $Y = sin(\pi X/2)$.

   a. What is the sample space of Y? In other words, what values can Y take. (2 pt)

   b. Find the pmf of Y. (4 pts)

EE 131A **Final Exam 2(25 pts; Open book; 90 min.)** **Winter 2013**

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1. All six faces of a "fair die" have equal probabilities of appearing. Suppose you suspect the face with the "six dots" does not appear with equal probability. You toss the die 180 times trying to detect any irregularity.

   a. If the die is fair, what is the ensemble average of the number of times the "six dots" will appear? (1 pt)

   b. Of course, in practice the actual number of times the "six dots" will appear will not be exactly equal to that ensemble average. Write down the analytical express for the probability that a fair die will have less than 20 and and more than 40 number of "six dots" appearing in 180 tosess? Do not evaluate this expression. (2 pts)

   c. Since the above analytical expression of the probability is too computationally complicated to evaluate, let us use the Central Limit Theorem to approximate this expression. What is the numerical value for this aproximate expression? Hint: We know for a Gaussian CDF, $\Phi(0) = 0.5$; $\Phi(1) = 0.8413$; $\Phi(2) = 0.9772$. (3 pts)

2. Let $\{X_1, X_2, X_3\}$ be a zero-mean random sequence with an autocorrelation function $R(X_i X_j) = E\{X_i X_j\} = a^{|i-j|}$, $i,\ j = 1, 2, 3$, where $0 < a < 1$.

   a. Under the mean-square (ms) error criterion, we vary $\{c_1, c_2\}$ such that $\epsilon^2(c_1, c_2) = E\{(X_3 - (c_1 X_1 + c_2 X_2))^2\}$ is minimized with $\epsilon^2_{min}(\hat{c}_1, \hat{c}_2) = min\{\epsilon^2(c_1, c_2)\} = E\{(X_3 - (\hat{c}_1 X_1 + \hat{c}_2 X_2))^2\}$. Find $\{\hat{c}_1, \hat{c}_2\}$. (2 pts). Find $\epsilon^2_{min}(\hat{c}_1, \hat{c}_2)$. (2 pts)

   b. Under the mean-square (ms) error criterion, we vary $\{c_1\}$ such that $\epsilon^2(c_1) = E\{(X_2 - c_1 X_1)^2\}$ is minimized with $\epsilon^2_{min}(\hat{c}_1) = min\{\epsilon^2(c_1)\} = E\{(X_2 - (\hat{c}_1 X_1)^2\}$. Find $\{\hat{c}_1\}$. (2 pts). Find $\epsilon^2_{min}(\hat{c}_1)$. (2 pts)

3. Let X and Y be two independent positive-valued r.v.'s with their pdf's given by $f_X(x) = 0,\ x < 0$ and $f_Y(y) = 0,\ y < 0$. Define the r.v. $Z = X \times Y$, where $\times$ is the regular multiplication (e.g., $3 \times 2 = 6$.) and is not the convolution operation. Find the pdf of $f_Z(z)$, $0 < z < \infty$. Hint: Start with $F_Z(z)$. (6 pts)

4. Let the random vector $X = [X_1, X_2]^T$ have a mean vector $\mu_X = [1, 2]^T$ and a covariance matrix

$$\mathbf{R}_X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

   Let $Y = AX$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

   Find the mean vector μ_Y and the covariance matrix $\mathbf{R}_Y$ of Y. (5 pts)

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Final Exam I Solutions

1\. a. i. $\int_0^1\int_0^1 bxy\,dx\,dy = b \times \frac{x^2}{2}\Big|_0^1 \times \frac{y^2}{2}\Big|_0^1 = \frac{b}{4} = 1 \Rightarrow b = 4.$

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ii. $f_X(x) = \int_0^1 f(x,y)\,dy = 4x\left(\frac{y^2}{2}\right)\Big|_0^1 = 2x, \quad 0 \le x \le 1.$

$f_Y(y) = 2y, \quad 0 \le y \le 1.$

iii. $f_X(x) f_Y(y) = (2x)(2y) = 4xy = f_{X,Y}(x,y), \quad 0 \le x \le 1, \; 0 \le y \le 1.$

Thus, X and Y are independent r.v.'s.

b. i. $\int_0^1 c\left[\int_0^y x\,dx\right] y\,dy = \frac{c}{2}\int_0^1 y\,(x^2)\Big|_0^y dy = \frac{c}{2}\int_0^1 y^3\,dy = \frac{c}{8}\,y^4\Big|_0^1 = \frac{c}{8} = 1$

$\Rightarrow c = 8.$

ii. $f_Y(y) = \int_0^y f(x,y)\,dx = 8y\int_0^y x\,dx = 8y\left(\frac{y^2}{2}\right)\Big|_0^y = 4y^3, \quad 0 \le y \le 1.$

$f_X(x) = \int_x^1 f(x,y)\,dy = 8x\int_x^1 y\,dy = 8x\left(\frac{y^2}{2}\right)\Big|_x^1 = 4x(1-x^2)$

$= 4x - 4x^3, \quad 0 \le x \le 1$

iii. Clearly $f_X(x) f_Y(y) = 4y^3(4x - 4x^3) \neq f_{X,Y}(x,y) = 8xy, \quad 0 \le x \le y \le 1.$

Thus, X and Y are NOT independent.

2\. a. Clearly, $C_r^{m+n} = C_3^7 = \frac{7!}{3!4!} = \frac{7\times6\times5}{3\times2\times1} = 35.$

b. $\sum_{k=0}^{3} C_k^4 C_{3-k}^3 = C_0^4 C_3^3 + C_1^4 C_2^3 + C_2^4 C_1^3 + C_3^4 C_0^3$

$= 1 + 4\times3 + \frac{4\times3}{2}\times3 + 4\times1 = 1+12+18+4 = 35.$

3\. a $F_{X|B}(x|B) = \frac{P(\{X \le x\} \cap \{X \le b\})}{P(X \le b)} = \begin{cases} P(X \le b)/P(X \le b), & \text{if } b \le x; \\ P(X \le x)/P(X \le b), & \text{if } x \le b, \end{cases}$

$= \begin{cases} 1, & b \le x \\ (x/a)/(b/a), & x \le b \end{cases} = \frac{x}{b}, \quad 0 \le x \le b.$

b. $f_{X|B}(x|B) = \frac{d}{dx} F_{X|B}(x|B) = \begin{cases} \frac{1}{b}, & 0 \le x \le b. \\ 0, & \text{elsewhere} \end{cases}$

4. a. Clearly, Y can only take the values of $\{0, 1, -1\}$.

b. $Y=0 \Leftrightarrow X \in \{2, 4, 6, \cdots = 2k, \ k = 1, 2, \cdots\}$

$Y = 1 \Leftrightarrow X \in \{1, 5, 9, \cdots = 4k+1, \ k = 0, 1, 2, \cdots\}$

$Y = -1 \Leftrightarrow X \in \{3, 7, 11, \cdots = 4k+3, \ k = 0, 1, 2, \cdots\}$.

$$P(Y=0) = \sum_{k=1}^{\infty} P(X=2k) = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{2k} = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^{k} = -1 + \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{k}$$

$$= -1 + \frac{1}{1-\frac{1}{4}} = -1 + \frac{1}{3/4} = -1 + \frac{4}{3} = \frac{1}{3}.$$

$$P(Y=1) = \sum_{k=0}^{\infty} P(X=4k+1) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{4k+1} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^{k}$$

$$= \frac{1}{2} \times \frac{1}{1-\frac{1}{16}} = \frac{1}{2} \times \frac{16}{15} = \frac{8}{15}$$

$$P(Y=-1) = \sum_{k=0}^{\infty} P(X=4k+3) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{4k+3} = \frac{1}{8} \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^{k}$$

$$= \frac{1}{8} \times \frac{1}{1-\frac{1}{16}} = \frac{1}{8} \times \frac{16}{15} = \frac{2}{15}.$$

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Final Exam 2 Solutions K. Yao

1. a. $= \mu =$ Average $= 180 \times \frac{1}{6} = 30$.

b. $\tilde{P} =$ Prob. a fair die with 180 tosses will have less than 20 and more then 40 numbers $= 1 - \sum_{k=20}^{40} C_k^{180} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{180-k}$.

c. Using CLT, with $\mu = 30$, $\sigma^2 = 180 \times \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = 25$, $\sigma = 5$.

$$\tilde{p} = 1 - P(20 \le X \le 40) = 1 - P\left(\frac{20-30}{5} \le \frac{X-30}{5} \le \frac{40-30}{5}\right)$$
$$= 1 - P(-2 \le Z \le 2) = 1 - (\Phi(2) - \Phi(-2))$$
$$= 1 - (0.9772 - (1 - 0.9772)) = 2 - 2 \times 0.9772 = 0.0456.$$

2 a. $R = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$; $R^{-1} = \frac{1}{1-a^2}\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}$; $r = \begin{bmatrix} a \\ a^2 \end{bmatrix}$; $Jr = \begin{bmatrix} a^2 \\ a \end{bmatrix}$

$$\hat{C} = \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \end{bmatrix} = R^{-1} J r = \frac{1}{1-a^2}\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}\begin{bmatrix} a^2 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ a \end{bmatrix} \Rightarrow \hat{c}_1 = 0,\ \hat{c}_2 = a.$$

$$\epsilon^2_{min}(\hat{c}_1, \hat{c}_2) = 1 - [a \ \ a^2] \frac{1}{1-a^2}\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}\begin{bmatrix} a \\ a^2 \end{bmatrix} = 1 - a^2.$$

b. $\hat{c} : \hat{c}_1$; $R = 1$, $R^{-1} = 1$; $r = a$. $\hat{c}_1 = a$.

$\epsilon^2_{min}(\hat{c}_1) = 1 - a^2$.

3. $$F_Z(z) = P(Z \le z) = P(X \cdot Y \le z) = P\left(X \le \frac{z}{Y}\right) = \int_0^\infty P\left(X \le \frac{z}{y}\right) f_Y(y)\, dy$$
$$= \int_0^\infty F_X\left(\frac{z}{y}\right) f_Y(y)\, dy = \int_0^\infty \int_{-\infty}^{z/y} f_X(t)\, dt\, f_Y(y)\, dy.$$

$f_Z(z) = \frac{d F_Z(z)}{dz} = \int_0^\infty f_X\left(\frac{z}{y}\right) \frac{1}{y} f_Y(y)\, dy$. Similarly, $f_Z(z) = \int_0^\infty f_Y\left(\frac{z}{x}\right) \frac{1}{x} f_X(x)\, dx$

4. $\mu_Y = E\{Y\} = A\mu_X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$$R_Y = E\{(Y-\mu_Y)(Y-\mu_Y)^T\} = E\{A(X-\mu_X)(X-\mu_X)^T A^T\}$$
$$= A R_X A^T = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ 18 & 25 \end{bmatrix}$$