**EE 131A** **Final Exam 1(25 pts; Closed book; 90 min.)** **Winter 2012**

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1. Consider an urn with objects marked $\{1, 2, \ldots, 9\}$. We take two objects out one at at a time with replacement. What is the probability that both objects have the same number? (3 pts)

2. The IQ of a randomly selected person is often assumed to have a normal distribution with a mean of 100 and a standard deviation of 15. Find the probability that a person has an IQ of:

   **a.** Above 140. (1 pt)

   **b.** Between 120 and 130. (2 pts)

   **c.** Find a value of $x$ such that 99% of the population has IQ of at least $x$. (2 pts)

   You can use the $\Phi(x)$ table given in the back to find the numbers in all three parts of this question.

3. Consider the r.v. $Y = 1/X$, where the r.v. $X$ is uniformly distributed on (0, 1).

   **a.** Find the cdf $F_Y(y)$ and its domain. (2 pts)

   **b.** Find the median of $Y$ denoted by $m$ and defined by $F_Y(m) = 1/2$. (1 pt)

   **c.** Find the pdf $f_Y(y)$ and its domain. (2 pts)

   **d.** Find the mean $\mu_Y = E\{Y\}$. (2 pts)

4. Consider a r.v. $X$ with a mean $\mu_X$ and a variance $\sigma_X^2$.

   **a.** Find the mean and variance of $-X$. (2 pts)

   **b.** Let the r.v. be defined by $Y = a\,X + b$. Find the constants $a$ and $b$ such that the r.v. $Y$ has zero mean and variance 1. (3 pts)

5. Consider a continuous r.v. $X$ with a pdf given by

$$f(x) = \begin{cases} (1/6)(x^3 + 1)\,, & 0 < x < 2\,, \\ 0\,, & \text{elsewhere}\,. \end{cases}$$

   Find $E\{X^n | X < 1\}$, where $n$ is a non-negative integer. (5 pts)

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1. 9 possible cases with ~~two~~ similar numbers out of 81 cases. $P=\frac{9}{81}=\frac{1}{9}$.

2. a. $P(X>140)=1-P(X\le 140)=1-\Phi\left(\frac{140-100}{15}\right)=1-\Phi(2.67)\simeq 0.004$

b. $P(120\le X\le 130)=\Phi\left(\frac{130-100}{15}\right)-\Phi\left(\frac{120-100}{15}\right)=\Phi(2)-\Phi(1.33)\simeq 0.07$

c. $P(X>x)=0.99$ or $P(X>x)=1-\Phi\left(\frac{x-100}{15}\right)=\Phi\left(\frac{100-x}{15}\right)=0.99$.

From the $\Phi(\cdot)$ table, $\frac{100-x}{15}=2.33$ yields $x\simeq 65$.

3. a. Since $f_X(x)=1$, $0<x<1$, then $F_X(x)=x$, $0<x<1$. Let $y=g(x)=\frac{1}{x}$, $1<y<\infty$.

$F_Y(y)=P(Y\le y)=P(\frac{1}{X}\le y)=P(\frac{1}{y}\le X)=1-P(X<\frac{1}{y})=1-F_X(\frac{1}{y})=1-\frac{1}{y}$, $1<y<\infty$

b. Median of $Y=m$ satisfies $F_Y(m)=\frac{1}{2}$.

$F_Y(m)=1-\frac{1}{m}=\frac{1}{2}$, or $m=2$.

c. $f_Y(y)=\frac{d}{dy}(-y^{-1})=\frac{1}{y^2}$, $1<y<\infty$.

d. $\mu_Y=E\{Y\}=\int_1^\infty y\times\frac{1}{y^2}dy=\int_1^\infty\frac{dy}{y}=\ln(y)\Big|_1^\infty=\infty-0=\infty$.

4. a. $E\{-X\}=-\mu_X$. $\sigma^2_{-X}=m_2(-X)-(m_1(-X))^2=m_2(X)-(-\mu_X)^2=\sigma_X^2$.

b. From $Y=aX+b$, we want $E\{Y\}=a\mu_X+b=0$, or $a=\frac{-b}{\mu_X}$ (1).

We also want $Var\{Y\}=E\{Y^2\}=E\{a^2X^2+2abX+b^2\}=a^2m_2(X)+2ab\mu_X+b^2=1$

Since $m_2(X)=\sigma_X^2+\mu_X^2$, then we have $a^2(\sigma_X^2+\mu_X^2)+2ab\mu_X+b^2=1$ (2).

Substitute (1) in (2) to yield $\frac{b^2}{\mu_X^2}(\sigma_X^2+\mu_X^2)-\frac{2b^2}{\mu_X}\mu_X+b^2=1$ or

$b^2=\frac{\mu_X^2}{\sigma_X^2}$ or $b=\pm\frac{\mu_X}{\sigma_X}$. Thus, two solutions are

1. $a=-\frac{1}{\sigma_X}$ and $b=\frac{\mu_X}{\sigma_X}$ or 2. $a=\frac{1}{\sigma_X}$ and $b=-\frac{\mu_X}{\sigma_X}$.

5. $f(u|X<1)=\frac{\frac{1}{6}(u^3+1)}{P(X<1)}$, $0<u<1$. $P(X<1)=\int_0^1\frac{1}{6}(x^3+1)dx=\frac{5}{24}$.

$$E\{X^n|X<1\}=\int_0^1 u^n\times\frac{4}{5}(u^3+1)du$$

$$=\frac{4}{5}\left[\frac{u^{n+4}}{n+4}+\frac{u^{n+1}}{n+1}\right]\Bigg|_{u=0}^{u=1}=\frac{4}{5}\times\frac{2n+5}{(n+4)(n+1)}.$$

**EE 131A** **Final Exam 2(25 pts; Open book; 90 min.)** **Winter 2012**

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1. Consider two r.v. $\{X, Y\}$ which are uniformly distributed in the unit disk of $x^2 + y^2 \leq 1$.

   **a.** Find the constant $c$ in

$$f_{X,Y}(x, y) = \begin{cases} c\,, \ x^2 + y^2 \leq 1\,, \\ 0\,, \ \text{elsewhere}\,. \end{cases}$$

(3 pts)

   **b.** Find the marginal pdf $f_X(x)\,,\ -1 \leq x \leq 1$. (2 pts)

   **c.** Find the marginal pdf $f_Y(y)\,,\ -1 \leq y \leq 1$. (1 pt)

   **d.** Are $X$ and $Y$ independent r.v.'s? (1 pt)

2. On a ten equation true-false test, a student guesses on every question independently and with equal probability of being right or wrong. Find the probability that the student answers:

   **a.** No question correctly. (2 pts)

   **b.** At least one question correctly. (2 pts)

   **c.** Exactly r questions correctly, $r = 0, 1, \ldots, 10$. (2 pts)

3. In the class lecture, given an r.v. $Y$, we have considered a linear m.s. estimation of $Y$ using $a\,X + b$. Now, consider a nonlinear estimation estimator $g(X)$ to estimate the given r.v. $Y$. Find the optimum estimator $\hat{g}(X)$ such that the m.s. estimation error $\epsilon^2 = E\{(Y - g(X))^2\}$ is minimized. Hint: Use the information of $f_{X,Y}(x, y) = f_{Y|X}(y|x)f_X(x)$ and that if $X = x$ is conditioned, then $g(X)$ a constant in the above equation. (4 pts)

4. We are told the following two pieces of information. A: The telephone arrivals at a company in one minute is Poisson distributed with an average rate of 4. B: Half of the calls are females and the other half are males, and all calls are independent of each other.

   Let $X$ be a r.v. that denotes the number of female calls in one minute and let $Y$ be the r.v. be the total number of calls in one minutes.

   **a.** Find the pmf $p_{X,Y}(j\,,k)\,, 0 \leq j \leq k\,, k = 0, 1, 2\,, \ldots$. Hint: Conditioned on $Y = k$, what is the distribution of $X$? (4 pts)

   **b.** Find the marginal pmf $p_X(j)\,,\ j = 0, 1\,, \ldots$. Hint: $\exp(a) = \sum_{m=0}^{\infty}(a^m)/(m!)$. (3 pts)

   **c.** What kind of r.v. is $X$? (1 pt)

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1a. We know $\iint_{x^2+y^2=1} c\,dx\,dy = 1$. using polar coord. $x = r\cos(\theta)$, $y = r\sin(\theta)$, $dx\,dy = r\,dr\,d\theta$ in the Gaussian problem. Then

$$1 = \iint_{x^2+y^2} c\,dx\,dy = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} c \cdot r\,dr\,d\theta = c \times 2\pi \int_{r=0}^{1} r\,dr = c \times 2\pi \times \frac{r^2}{2}\Big|_{r=0}^{1} = c \times \pi$$

Thus, $c = 1/\pi$.

b. From $x^2+y^2=1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$. Thus, $f_X(x) = \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy = \frac{2}{\pi}\sqrt{1-x^2}$, $-1 \le x \le 1$.

c. By symmetry, $f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}$, $-1 \le y \le 1$.

d. Since $f_X(y) f_Y(y) = \frac{4}{\pi^2}(\sqrt{1-x^2})(\sqrt{1-y^2}) \ne f_{X,Y}(x,y) = c$, $x^2+y^2 \le 1$, then $X$ and $Y$ are not indep. r.v.'s.

2. a. Prob. no answer correctly $= \frac{1}{2^{10}}$. b. Prob. at least one corret $= 1 - \frac{1}{2^{10}}$.

c. Prob. exactly $r$ correct $= \frac{C_r^{10}}{2^{10}}$.

3. $\epsilon^2 = E\{(Y-g(X))^2\} = \iint (y-g(x))^2 f_{X,Y}(x,y)\,dx\,dy = \int f_X(x)\left[\int (y-g(x))^2 f_{Y|X}(y|x)\,dy\right]dx$

Since $f_X(x) \ge 0$, for each conditioned $x$, then $g(x)$ is a constant. But the opt. est. of a r.v. by a constant in the m.s. sense is

$$\mathrm{Min}_{\text{fixed } x}\left[\int (y-g(x))^2 f_{Y|X}(y|x)\,dx\right] = \int\left(y - \int y' f_{Y|X}(y'|x)\,dy'\right)^2 f_{Y|X}(y|x)\,dy$$

Thus, the opt. $\hat{g}(x) = \int y' f_{Y|X}(y'|x)\,dy' = E\{Y \mid X=x\}$ = Cond. exp. of $Y$ given $X=x$.

4. a. Conditioned on $Y=k$, $P(X=j \mid Y=k) = C_j^k (0.5)^j (0.5)^{k-j}$, $0 \le j \le n$.

and $P(Y=k) = \frac{e^{-4}4^k}{k!}$, $k = 0, 1, \cdots$

Thus, $P_{X,Y}(j,k) = P(X=j \mid Y=k)\,P(Y=k) = \frac{e^{-4}(\frac{1}{2})^k 4^k}{j!\,(k-j)!} = \frac{e^{-4}2^k}{j!\,(k-j)!}$, $0 \le j \le k$, $k = 0, 1, \cdots$

b. $P_X(j) = \sum_{k=j}^{\infty} \frac{2^k e^{-4}}{j!\,(k-j)!} = \frac{e^{-4}}{j!}\sum_{k=j}^{\infty}\frac{2^k}{(k-j)!}$. Let $m = k-j$.

$$= \frac{e^{-4}}{j!}\sum_{m=0}^{\infty}\frac{2^{j+m}}{m!} = e^{-4}2^j \sum_{m=0}^{\infty}\frac{2^m}{m!} = e^{-4}2^j e^2 = \frac{e^{-2}2^j}{j!},$$

$j = 1, 2, \cdots$

c. $X$ is a Poisson r.v. with $\lambda = 2$.