EE 131A Fall 2002 Midterm Exam. October 29, 2002.

Instructor: Rick Wesel

70 pts

Time: 110 Minutes

Coverage: Sections 2.1-2.6 and 3.1-3.6 of Leon Garcia

Name	SOLUTIONS
Student ID Number	
Email Address	

Problem Number	Points	Possible Points
1		12
2		12
3		10
4		12
5		12
6		12
Total		70

1. *A Foolish Thief.* Joe is a fool with probability 0.6, a thief with probability 0.7, and neither with probability 0.25

   a. (6 pts) Determine the probability that he is a fool or a thief, but not both. **We know the following:** $P[F]=0.6,\ P[T]=0.7,\ P\left[(F\cup T)^c\right]=0.25$.

   The probability in question is $P\left[(F\cup T)\cap\left((F\cap T)^c\right)\right]$. **Note that** $(F\cup T)=\{(F\cap T)\}\cup\left\{(F\cup T)\cap\left((F\cap T)^c\right)\right\}$, **where the sets in braces are mutually exclusive. Thus** $P[(F\cup T)]=P[(F\cap T)]+P\left[(F\cup T)\cap\left((F\cap T)^c\right)\right]$ **and then** $P\left[(F\cup T)\cap\left((F\cap T)^c\right)\right]=P[(F\cup T)]-P[(F\cap T)]$. **So the whole thing boils down to finding** $P[(F\cup T)]$ **and** $P[(F\cap T)]$. **Well,** $P[(F\cup T)]=1-P\left[(F\cup T)^c\right]=0.75$ **and** $P[(F\cap T)]=P[F]+P[T]-P[F\cup T]=0.6+0.7-0.75=0.55$. **So finally, we have** $P\left[(F\cup T)\cap\left((F\cap T)^c\right)\right]=0.75-0.55=0.2.$

   b. (6 pts) Determine the conditional probability that he is a thief, given that he is not a fool. **The probability in question is** $P\left[T\mid F^c\right]$. **Using the definition of conditional probability,** $P\left[T\mid F^c\right]=\dfrac{P\left[T\cap F^c\right]}{P\left[F^c\right]}$.

   $P\left[T\cap F^c\right]=P[T]-P[T\cap F]=0.7-0.55=0.15$, **and** $P\left[F^c\right]=1-P[F]=0.4$. **Thus we can finally finish this foolish question with** $P\left[T\mid F^c\right]=\dfrac{P\left[T\cap F^c\right]}{P\left[F^c\right]}=\dfrac{0.15}{0.4}=0.375$.

2. *Flush.* A deck of 52 cards has four "suits": hearts, spades, clubs, and diamonds. There are thirteen cards of each suit. Nine of these are numbered with the numbers 2 through 10. The remaining four cards in each suit are called the Ace, King, Queen, and Jack. You draw seven cards from a perfectly shuffled deck.

   a. (4 pts) What is the probability you draw a "Spade Flush" in which all seven cards are from the spade suit?

      This probability is $\frac{\binom{13}{7}}{\binom{52}{7}} = \frac{1716}{133784560} = 1.28266 \times 10^{-5}$.

   b. (4 pts) What is the probability of drawing a "Flush" in which all seven cards are from the same suit.
      There are four mutually exclusive events (one for each suit), each with the probability described above. So the probability is
      $4 \times 1.28266 \times 10^{-5} = 5.13 \times 10^{-5}$.

   c. (4 pts) What is the probability you draw a "Royal Flush", which consists of the Ace, King, Queen, Jack, and Ten of the same suit plus any two other cards.

      The number of Royal Flush hands is $4 \times \binom{47}{2}$**, where the 4 selects the suit that is shared by the Ace, King, Queen, Jack, and Ten and the choose term selects the remaining two cards. Thus the probability is question is** $\frac{4 \times \binom{47}{2}}{\binom{52}{7}}$.

3. *Memoryless.* Let X be a discrete random variable described by a geometric PMF.

a. (6 pts) Given that the experimental value of X is greater than integer y, derive the conditional probability for x given that $X > y$, i.e. find $P_X[X = x \mid X > y]$.

This is a conditional probability question.

$$P_X[X = x \mid X > y] = \frac{P[(X = x) \cap (X > y)]}{P[(X > y)]}$$

$$= \begin{cases} 0 & x \le y \\ \dfrac{(1-p)^{x-1} p}{\sum_{k=y+1}^{\infty} (1-p)^{k-1} p} & x > y \end{cases}$$

$$= \begin{cases} 0 & x \le y \\ \dfrac{(1-p)^{x-1} p}{(1-p)^{y}} & x > y \end{cases}$$

$$= \begin{cases} 0 & x \le y \\ (1-p)^{x-y-1} p & x > y \end{cases}$$

b. (4 pts) Given that the experimental value of X is greater than integer y, derive the conditional probability for $x - y$ given that $X > y$, i.e. find $P_X[(X-y) = (x-y) \mid X > y]$.

This is an "integers-to-integers" derived distribution. Once the notation is understood, it is straightforward. The random variable has exactly the PMF derived above.

$$P_X[X - y = x - y \mid X > y] = P_X[X = x \mid X > y]$$

$$\begin{cases} 0 & x \le y \\ (1-p)^{x-y-1} p & x > y \end{cases}$$

Because the original notation (corrected above) was not precise, full points were also given for the following (popular) alternative interpretation of the original problem statement.

$$P_X[X = x - y \mid X > y] = \begin{cases} 0 & x \le 2y \\ (1-p)^{x-2y-1} p & x > 2y \end{cases}$$

4. *Scones*. Kevin has a fondness for chocolate chip scones from the bakery across the street. However, he is particular that the number of chocolate chips in each scone be at least 50. It turns out that the number of chips in a scone follows the Poisson probability law $P_K(k) = \frac{\alpha^k}{k!} e^{-\alpha}$ with the average number of chips in a scone being 50.

a. (4 pts) What is the probability a scone will have exactly 50 chocolate chips?

$$\alpha = 50,\ k = 50. \text{ So } P_K(50) = \frac{50^{50}}{50!} e^{-50}$$

b. (4 pts) With each bite, Kevin eats exactly one fiftieth $\left(\frac{1}{50}\right)$ of a scone. The number of chips in a bite is also Poisson. What is the probability that Kevin eats no chocolate chips in a particular bite?

$$\text{Now } \alpha = 1,\ k = 0. \text{ So } P_K(0) = e^{-1}$$

c. (4 pts) If Kevin eats a bite with no chocolate chip, the scone is declared defective and thrown on the floor of the minivan. Find the probability that a scone is declared defective at some point during the 50 bites required for its consumption.
It is easier to find the probability that the scone is never defective. This is simply $\left(1-e^{-1}\right)^{50}$. The probability of interest is thus $1-\left(1-e^{-1}\right)^{50}$.

5. *Darts*. The cumulative distribution for the radius at which a dart arrives on a unit-radius dartboard is simply

$$F_R(r) = \begin{cases} 0 & r \le 0 \\ r^2 & 0 < r \le 1 \\ 1 & r > 1 \end{cases}$$

a. (4 pts) What is the probability a dart lands at a radius that lies in the interval (0.25,0.5]?

$$P[(0.25,0.5]] = F_R(0.5) - F_R(0.25)$$
$$= 0.25 - 0.125 = 0.125.$$

b. (4 pts) What is the probability a dart lands with a radius equal to 0.5 exactly?

For a continuous random variable, the probability of any single point is zero.

c. (4 pts) How large should the radius of the center circle ("Bull's Eye") be if the probability of landing in the Bull's Eye is to be 0.01?

For $r = 0.1$, $F_R(0.1) = 0.01$

6. *Derived Distribution.* Let X be a uniform random variable on the unit interval:

$$f_X(x)=\begin{cases}1 & x\in(0,1)\\ 0 & \text{elsewhere}\end{cases}$$

a. (4 pts) Find $F_X(x)$.

$$F_X(x)=\begin{cases}0 & x\le 0\\ x & x\in(0,1]\\ 1 & x>1\end{cases}$$

b. (4 pts) Define $Y=-\ln(X)$. Note that Y is always positive. Find $F_Y(y)$.

$\{Y\le y\}$ is equivalent to $\{-\ln X\le y\}$ which is the same as $\{X\ge e^{-y}\}$. Thus $F_Y(y)=1-F_X(e^{-y})$ for $e^{-y}\in(0,1]$ which is $y\in[0,\infty)$. For $y<0$, $F_Y(y)=0$. Summarizing:

$$F_Y(y)=\begin{cases}0 & y<0\\ 1-e^{-y} & y\ge 0\end{cases}$$

c. (4 pts) Find $f_Y(y)$.

$$f_Y(y)=\frac{dF_Y(y)}{y}=\begin{cases}0 & y<0\\ e^{-y} & y\ge 0\end{cases}$$