EE 131A Practice Midterm Problems Solutions

Probability

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1. Problem 1. (total: 20pts) From a group of 5 women and 7 men, a committee is to be formed by picking 5 members randomly from this group of 12 individuals:

   (a) (5pts) What is the probability that the committee has exactly 2 women and 3 men?

   (b) (10pts) Two men, Mr. Jones and Mr. Chen, are sworn enemies. What is the probability that the committee has two women and 3 men, but not both Mr Jones and Mr. Chen?

   (c) (5pts) Given that the randomly picked committee has 2 women and 3 men, what is the conditional probability that the committee does not have both Mr Jones and Mr. Chen?

   **Solution:**

   (a) Let A be the event that the committee has exactly 2 women and 3 men, we have,

$$P[A] = \frac{\binom{5}{2}\binom{7}{3}}{\binom{12}{5}} = \frac{175}{396}$$

   (b) Let B be the event that the committee has two women and 3 men, but not both Mr Jones and Mr. Chen. We have,

$$P[B] = \frac{\binom{5}{2}(\binom{5}{3} + \binom{2}{1}\binom{5}{2})}{\binom{12}{5}} = \frac{25}{66}$$

   (c)

$$P[B|A] = \frac{P[B \cap A]}{P[A]} = \frac{P[B]}{P[A]} = \frac{6}{7}$$

2. Problem 2 (total: 25 pts) A bin contains three different types of disposable flashlisghts. The probability that a type 1 flashlisght will give over 100 hours of use is 0.7; the corresponding probabilities for type 2 and and 3 flashlisghts being 0.4 and 0.3, respectively. Suppose that 20% of the flashlights in the bin are type 1, 30% are type 2, and 50% are type 3.

   (a) (10pts) What is the probability that a randomly chosen flashlight will give more than 100 hours of use?

   (b) (15 pts) Given that a randomly picked flashlight lasts over 100 hours, what is the conditional probability that that it is of type j flashlight for j= 1, 2, and 3?

   **Solution:**

(a) Let $A_k$ be the event that the flashlight is of type $k$, and B be the event that the flashlight will give more than 100 hrs of use, we have,

$$\begin{aligned} P[B] &= P[B|A_1]P[A_1] + P[B|A_2]P[A_2] + P[B|A_3]P[A_3] \\ &= 0.2 \cdot 0.7 + 0.3 \cdot 0.4 + 0.5 \cdot 0.3 = 0.41 \end{aligned}$$

(b)

$$\begin{aligned} P[A_1|B] &= \frac{P[B|A_1]P[A_1]}{P[B]} \\ &= \frac{14}{41} \end{aligned}$$

(b)

$$\begin{aligned} P[A_2|B] &= \frac{P[B|A_2]P[A_2]}{P[B]} \\ &= \frac{12}{41} \end{aligned}$$

(b)

$$\begin{aligned} P[A_3|B] &= \frac{P[B|A_3]P[A_3]}{P[B]} \\ &= \frac{15}{41} \end{aligned}$$

3. Problem 3. (15pts) A laboratory blood test is 95% effective in detecting a certain disease when it is in fact present. (That is, if a person with the disease is tested then with probability 0.95 the test will come out positive). However, the test also yields a false positive result for 1% of the healthy persons tested. (That is, if a healthy person is tested, then with probability 0.01, the test will imply he or she has the disease, i.e., the test will be positive.)

   If 0.5% of the population actually has the disease, what is the probability a randomly tested person actually has the disease given that the test result is positive.
   **Solution:** Let A = {The tested person has disease}.
   T = {Test result is positive},

   We have,

$$\begin{aligned} P[A|T] &= \frac{P[T|A]P[A]}{P[T|A]P[A] + P[T|A^c]P[A^c]} \\ &= \frac{0.95 \cdot 0.005}{0.95 \cdot 0.005 + 0.01 \cdot 0.995} \\ &= \frac{95}{294} \end{aligned}$$

4. Problem 4. (total: 30 pts) The game of craps is played as follows: A player rolls two fair dice. If the sum of the dice is either 2, 3, or 12, the player loses; if the sum is either a 7 or an 11, he or she wins. If the outcome is anything else, the player continues to roll the two dice until he or she rolls either the initial outcome or a 7. If the 7 comes first, the player loses; whereas if the initial outcome reoccurrs before the 7, the player wins.

   (a) (5 pts) What is the probability of winning on the first roll of the two dice?

   (b) (5 pts) What is the probability of losing on the first roll of the two dice?

   (c) (8 pts) Suppose that the first roll of the two dice yields a sum of 5. What is the probability that the player wins? [Hint: use a geometric law]

   (d) (7 pts) Suppose that the first roll of the two dice yields a sum of 6. What is the probability that the player wins? [Hint: use a geometric law]

   (e) (5 pts) Provide a strategy (not necessarily a complete solution) for computing the winning probability of a player at the game of craps.

   **Solution:** let us denote $X_i$ be to the sum of rolling two fair dice. For example: $P[X_i = 5] = P[\{1,4\},\{2,3\},\{3,2\},\{4,1\}] = \frac{4}{36}$, then full table of its pmf could be found as follows:

| $X_i$=k | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P[X_i = k]$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{35}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

   (a) The probability of wining on the first roll of the two dice is then given by:

$$P[\text{winning on the first roll}] = P[X_1 = 7] + P[X_1 = 11] = \frac{2}{9}$$

   (b) The probability of losing on the first roll of the two dice is then given by:

$$P[\text{losing on the first roll}] = P[X_1 = 2] + P[X_1 = 3] + P[X_1 = 12] = \frac{1}{9}$$

   (c) Given $X_1 = 5$ , then according to the pmf of $X_i$ , we can write the probability of winning in the nth round:

$$P[\text{wins at nth round}] = (\frac{4}{36})(1 - \frac{4}{36} - \frac{6}{36})^{n-1}$$

then the total probability of winning is given by:

$$P[\text{winning}] = \sum_{k=1}^{\infty}(\frac{26}{36})^k \frac{4}{36} = \frac{1}{1-\frac{26}{36}}\frac{4}{36} = \frac{2}{5}$$

Also we could solve it recursively by:

$$P[winning] = \frac{4}{36} + \frac{26}{36}P[winning]$$

(d) Given $X_1 = 6$ , then according to the pmf of $X_i$ , we can write the probability of winning in the nth round:

$$P[\text{wins at nth round}] = (\frac{5}{36})(1 - \frac{5}{36} - \frac{6}{36})^{n-1}$$

then the total probability of winning is given by:

$$P[\text{winning}] = \sum_{k=1}^{\infty} (\frac{25}{36})^k \frac{5}{36} = \frac{1}{1 - \frac{25}{36}} \frac{5}{36} = \frac{5}{11}$$

Also we could solve it recursively by:

$$P[winning] = \frac{5}{36} + \frac{25}{36} P[winning]$$

(e) Similarly we can compute the probability of winning as follows:

$$P[winning|X_1 = 4] = \frac{1}{3}$$

$$P[winning|X_1 = 8] = \frac{5}{11}$$

$$P[winning|X_1 = 9] = \frac{2}{5}$$

$$P[winning|X_1 = 10] = \frac{1}{3}$$

therefore the wining probability of a player at the game of craps is given by:

$$\begin{aligned} P[\text{winning}] &= \sum_{k=2}^{12} P[\text{winning}|X_1 = k] P[X_1 = k] \\ &= \frac{8}{36} + (\frac{1}{3} \times \frac{3}{36} + \frac{5}{11} \times \frac{5}{36} + \frac{2}{5} \times \frac{4}{36}) \times 2 \\ &= \frac{244}{495} \\ &= 0.4929 \end{aligned}$$

5. Problem 5. A binary information source (e.g.,a document scanner) generates very long strings of 0s followed by occasional 1s. Suppose that symbols are independent and that $p = P[symbol = 0]$ is very close to one. Consider the following scheme for encoding the run X of 0s between consecutive 1s:

   If $X = n$, express n as a multiple of an integer $M = 2^m$ and a remainder r, that is, find k and r such that $n = kM + r$, where $0 \leq r < M - 1$;

   The binary codeword for n then consists of a prefix consisting of k 0s followed by a 1, and a suffix consisting of the m-bit representation of the remainder r. The decoder can deduce the value of n from this binary string.

a) Find the probability that the prefix has k zeros, assuming that $p^M = 1/2$.

b) Find the average codeword length when $p^M = 1/2$.

c) Find the compression ratio, which is defined as the ratio of the average run length to the average codeword length when $p^M = 1/2$.

**Solution**:

a)

$$\begin{aligned} P = & \sum_{kM}^{kM+M-1} p^n(1-p) \\ = & p^{kM}(1-p^M) \\ = & (\frac{1}{2})^k(1-\frac{1}{2}) \\ = & (\frac{1}{2})^{k+1} \end{aligned}$$

b)

$$\begin{aligned} E[\text{length}] = & E[k] + 1 + m \\ = & \sum_{k=0}^{\infty} k\frac{1}{2}^{k+1} + 1 + m \\ = & m + 2 \end{aligned}$$

c)

$$\begin{aligned} E[\text{Run lengthz}] = & \sum_{0}^{\infty}(n+1)p^n(1-p) \\ = & (1-p)\frac{d}{dp}\sum_{n=0}^{\infty} p^{n+1} \\ = & \frac{1}{1-p} \end{aligned}$$

$$\text{Compress Ratio} = \frac{\frac{1}{1-p}}{m+2} = \frac{1}{(1-p)(m+2)}$$

6. Problem 6. In a game of poker five cards are picked at random from a deck of 52 cards. Note that a deck of cards has 13 denominations (or kinds) (namely, they are ordered as Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, and King) and 4 suits (namely, Hearts, Spades, Clubs, and Diamonds). What is the probability of being dealt

   (a) A full house? That is, when the cards have denominations a, a, a, b, b, where a and b are distinct. So, 10 of Spades, 10 of Hearts, 10 of Diamonds, Queen of Hearts and Queen of Clubs will comprise a full house.

(a) A Straight? This occurs when the cards have distinct conescutive denominations but not all of the same suit. So Ace of Spades, 2 of Hearts, 3 of Spades, 4 of Spades, and 5 of Spades will comprise a Straight. Note that Ace can be regarded as both the least or the greatest value, so 10 of Clubs, Jack of Clubs, Queen of Diamonds, King of Diamonds, and Ace of Diamonds will also comprise a Straight.

**Solutions:**

(a) There are $\binom{52}{5}$ total number of hands you can get with equal probability. To calculate how many of them are full house hands, we need to first choose a and b from 13 possible denominations and then choose a suit for each card:

$$P(\textit{full house}) = \frac{13 \times 12 \times \binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

(b) To calculate the number of straights, we choose the starting point of the straight which can be one of Ace, 2, 3, ..., 10, and then find the possible combination of suits except the ones where all the suits are the same$(4^5 - 4)$:

$$P(\textit{straight}) : \frac{10 \times (4^5 - 4)}{\binom{52}{5}}$$

7. Problem 7. Suppose you roll two fair dice. If the sum is greater than or equal to 10 you stop, but if the sum is less than 10 you roll the two dice one more time and then stop (so you either roll once or twice). If you stop the game with a sum less than 10, you lose \$10, but if you stop the game with a sum greater than or equal to 10, you win 3 times the amount of the sum in dollars.

   For example, (i) in one scenario in your first roll you might get a sum of 11, then you stop and you win \$33. (ii) in another scenario, in your first roll you might get a sum of 3, then you are forced to roll again, and say you get a sum of 12 in your second roll; then you win \$36, (iii) in yet another scenario, you might get a sum of 5 in your first roll, then roll again, and might get a sum of 4; then you lose \$10.

   (a) What is the probability of stopping after the first roll. What about the probability of having to roll twice? Hint: You may first want to calculate the probability of getting a sum of x in any roll of two fair dice, for x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

   (b) What is the probability of stopping with a sum of 11 (12)? What is the probability of stopping with a sum of 3?

   (c) If a player stops with a sum of 11 (or 12), what is the probability they have rolled twice? (Hint: Use Bayes Law).

   (d) What is the probability the player has rolled twice if they stopped with a sum of 3?

   (e) What is the expected value of your money winnings in this game?

**Solutions:**

(a) Let $S$ be the sum in a toss of two dice and $N_T$ the number of tosses:

$$P(N_T = 2) = P(S \geq 10) = \sum_{i=10}^{12} P(S = i) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{6}$$
$$P(N_T = 1) = 1 - P(N_T = 2) = \frac{5}{6}$$

(b) If $R$ is the final result sum:

$$P(R = 11) = P(S = 11) + P(N_T = 2) \times P(S = 11) = \frac{2}{36} + \frac{5}{6} \times \frac{2}{36} = \frac{11}{108}$$
$$P(R = 12) = P(S = 12) + P(N_T = 2) \times P(S = 12) = \frac{1}{36} + \frac{5}{6} \times \frac{1}{36} = \frac{11}{216}$$
$$P(R = 3) = P(N_T = 2) \times P(S = 3) = \frac{5}{6} \times \frac{2}{36} = \frac{5}{108}$$

(c) Using Bayes' Rule:

$$P(N_T = 2|R = 11) = \frac{P(R = 11|N_T = 2)P(N_T = 2)}{P(R = 11)} = \frac{\frac{2}{36} \times \frac{5}{6}}{\frac{11}{108}} = \frac{5}{11}$$

$$P(N_T = 2|R = 12) = \frac{P(R = 12|N_T = 2)P(N_T = 2)}{P(R = 12)} = \frac{\frac{1}{36} \times \frac{5}{6}}{\frac{11}{216}} = \frac{5}{11}$$

(d) $P(N_T = 2|R = 3) = \dfrac{P(R = 3|N_T = 2)P(N_T = 2)}{P(R = 3)} = \dfrac{\frac{2}{36} \times \frac{5}{6}}{\frac{5}{108}} = 1$

Which means if the result sum is 3 we have tossed twice for sure.

(e) Let $W$ be the winnings from the game:

$$E[W] = P(R < 10) \times (-10) + \sum_{i=10}^{12} P(R = i) \times (3i)$$

$$P(R = 10) = P(S = 10) + P(N_T = 2) \times P(S = 10) = \frac{3}{36} + \frac{5}{6} \times \frac{3}{36} = \frac{33}{216}$$

Therefore:

$$E[W] = \frac{5}{6}.\frac{5}{6}(-10) + \frac{33}{216} \times 30 + \frac{22}{216} \times 33 + \frac{11}{216} \times 36 = \frac{17}{6}$$

8. Problem 8. A man has 5 coins in his pocket. Two are double-headed, one is double-tailed, and two are normal. The coins cannot be distinguished unless one looks at them.

(a) The man shuts his eyes, chooses a coin at random, and tosses it. What is the probability that the lower face of the coin is heads?

(b) He opens his eyes and sees that the upper face of the coin is a head. What is the probability that the lower face is a head.

(c) He shuts his eyes again, picks up the same coin, and tosses it again. What is the probability that the lower face is a head?

(d) He opens his eyes and sees that the upper face is a head. What is the probability that the lower face is a head?

**Solutions:**

Let $A$ denote the event that he picks a double-headed coin, $B$ denote the event that he picks a normal coin, and $C$ be the event that he picks the double-tailed coin. Let $H_{L_i}$ (and $H_{L_i}$) denote the event that the lower face (and the upper face) of the coin on the $i$th toss is a head.

(a)

$$\begin{aligned} P(H_{L_1}) &= P(H_{L_1}|A)P(A) + P(H_{L_1}|B)P(B) + P(H_{L_1}|C)P(C) \\ &= 1 \times \frac{2}{5} + \frac{1}{2} \times \frac{2}{5} + 0 \times \frac{1}{5} = \frac{3}{5} \end{aligned}$$

(b)

$$P(H_{L_1}|H_{U_1}) = \frac{P(H_{L_1} \cap H_{U_1})}{P(H_{U_1})} = \frac{2/5}{3/5} = \frac{2}{3}$$

(c)

$$\begin{aligned} P(H_{L_2}|H_{U_1}) &= \frac{P(H_{L_2} \cap H_{U_1})}{P(H_{U_1})} \\ &= \frac{P(H_{L_2} \cap H_{U_1}|A)P(A) + P(H_{L_2} \cap H_{U_1}|B)P(B) + P(H_{L_2} \cap H_{U_1}|C)P(C)}{P(H_{U_1})} \\ &= \frac{1 \times \frac{2}{5} + \frac{1}{4} \times \frac{2}{5} + 0 \times \frac{1}{5}}{3/5} = \frac{5}{6} \end{aligned}$$

(d)

$$\begin{aligned} P(H_{L_2}|H_{U_1} \cap H_{U_2}) &= \frac{P(H_{L_2} \cap H_{U_1} \cap H_{U_2})}{P(H_{U_1} \cap H_{U_2})} \\ &= \frac{1 \times \frac{2}{5} + 0 \times \frac{2}{5} + 0 \times \frac{1}{5}}{1 \times \frac{2}{5} + \frac{1}{4} \times \frac{2}{5} + 0 \times \frac{1}{5}} = \frac{4}{5} \end{aligned}$$

9. Problem 9. A Christmas fruitcake has Poisson-distributed independent number of sultana raisins, iridescent red cherry bits, and radioactive green cherry bits with respective averages 48, 24, and 12 bits per cake. Suppose you politely accept 1/12 of a slice of the cake.

   (a) What is the probability that you get lucky and get no green bits in your slice?

   (b) What is the probability that you get really lucky and get no green bits and two or fewer red bits in your slice?

   (c) What is the probability that you get extremely lucky and get no green or red bits and more than five raisins in your slice?

   **Solution:**

   (a) Let S, R, and G be the number of sultana raisins, red bits, and green bits respectively. Each one of these variable has a poisson distribution with $\alpha = 48, 24, 12$. If we randomly pick 1/12 of the cake the number of bits in the slice will now have poisson distributions with averages of $\alpha = 4, 2, 1$ which are 1/12 of the average number of bits in the entire cake. So $S_s$, $R_s$, and $G_s$ which are the number of bits in the slice will have the following probability distributions:

   $$P(S_s = k) = \frac{4^k}{k!}e^{-4}$$
   $$P(R_s = k) = \frac{2^k}{k!}e^{-2}$$
   $$P(G_s = k) = \frac{1^k}{k!}e^{-1}$$

   Now the probability that we get no green bits is equal to:
   $P(G_s = 0) = e^{-1}$

   (b)

   $P(G_s = 0 \text{ and } R_s \leq 2) = P(G_s = 0)\sum_{k=0}^{2} P(R_s = k) = e^{-1}(e^{-2} + \frac{2}{1}e^{-2} + \frac{4}{2}e^{-2}) = 5e^{-1}e^{-2} = 5e^{-3}$

   (c)

   $P(G_s = 0 \text{ and } R_s = 0 \text{ and } S_s > 5) = P(G_s = 0)P(R_s = 0)\left(1 - \sum_{k=0}^{5} P(S_s = k)\right) = e^{-1}e^{-2}\left(1 - (e^{-4} + \frac{4}{1!}e^{-4} + \frac{4^2}{2!}e^{-4} + \frac{4^3}{3!}e^{-4} + \frac{4^4}{4!}e^{-4} + \frac{4^5}{5!}e^{-4})\right)$