Midterm Exam Solution

This exam has **7 problems** (the last one is a bonus problem).
There are **8 pages** in total.
You have **105 min**.

Only this booklet and your **two** sheets of notes should be on your desk.

Write your answer to each question in the space provided.

Use the backs of pages for intermediate calculations.

Show your reasoning clearly for partial credit.

Justify your results.

Read the problem statements carefully.

Problem 1 *(20 pts)*
Let A denote the event that Ms. Han attends Thursday's class, and B denote the event that Mr. Lee attends Thursday's class. Given Mr. Lee attends the class, the probability that A happens is 0.5. The probability that Mr. Lee attends Thursday's class is 0.2 and the probability that at least one of them attend the class is 0.8.

(a) (*10 pts*) Find the probability that (i) Ms. Han and Mr. Lee both attend Thursday's class. (ii) Ms. Han attends Thursday's class.

(b) (*5 pts*) Are events A and B independent? Are they mutually exclusive?

(c) (*5 pts*) Find the probability that neither of them attend Thursday's class and the probability that Mr. Lee attends class given Ms. Han doesn't attend Thursday's class.

Solution: The problem tells that $P(A \mid B) = 0.5$, $P(B) = 0.2$ and $P(A \cup B) = 0.8$

(a) (i) (4 pts) $P(A \cap B) = P(A \mid B)P(B) = 0.1$

(ii) (6 pts) Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, so

$$P(A) = P(A \cup B) - P(B) + P(A \cap B) = 0.8 - 0.2 + 0.1 = 0.7$$

(b) (i) (2.5 pts) $P(A \mid B) \neq P(A)$, so they are not independent.

(ii) (2.5 pts) $P(A \mid B) \neq 0$, so they are not mutually exclusive.

(c) (i) (2 pts) $P(A^{\mathsf{c}} \cap B^{\mathsf{c}}) = P((A \cup B)^{\mathsf{c}}) = 1 - P(A \cup B) = 0.2$

(ii) (3 pts) $P(B \mid A^{\mathsf{c}}) = \dfrac{P(B \cap A^{\mathsf{c}})}{P(A^{\mathsf{c}})} = \dfrac{P(A \cup B) - P(A)}{1 - P(A)} = \dfrac{0.8 - 0.7}{0.3} = \dfrac{1}{3}$

Problem 2 *(20 pts)*

(a) (*10 pts*) A space craft has 20,000 components. The probability of any one component being defective is 10^{-4}. The mission will be in danger if five or more components become defective. Find the probability of such an event. Use Poisson approximation.

(b) (*10 pts*) Jones figures that the total number of thousands of miles that an auto can be driven before it would need to be junked is an exponential random variable with parameter $\lambda = \dfrac{1}{20,000}$, i.e.,

$$P[\text{the car is not junked at the first } m \text{ miles}] = e^{-\lambda m}$$

Smith bought the car as a new car, and Smith has already driven the car for 10 thousand miles. Jones expects the car to run for another 20 thousand miles. What is the probability that Jones' expectations will be met?

That is, let A be the event that the car is not junked in the first 10 thousand miles. Let B be the event that Jones is able to drive the car for another 20 thousand miles. What is the probability $P[B \mid A]$?

Solution:

(a) Using Poisson approximation, the parameter $\alpha = 20000 \times 10^{-4} = 2$. Let the number of defective components be a random variable X, then $X \sim \textbf{Poisson}(2)$

$$\begin{aligned} P(\text{the mission is in danger}) &= P(X \geq 5) = 1 - \sum_{i=0}^{4} P(X = i) \\ &= 1 - e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right) \\ &= 1 - \frac{7}{e^2} \approx 0.052653 \end{aligned}$$

(b) $P(A) = e^{-10000\lambda} = e^{-1/2}$, $P(B) = e^{-(10000+20000)\lambda} = e^{-3/2}$, $P(A \cap B) = P(B)$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = e^{-1} \approx 0.367879$$

Problem 3 *(25 pts)*
A bowl contains twenty cherries, exactly fifteen of which have had their stones removed. A greedy pig eats two whole cherries, picked at random, without remarking on the presence or absence of stones.

(a) (*10 pts*) What is the probability that the pig eats k cherries with stones? ($k = \{0, 1, 2\}$)

(b) (*5 pts*) Now a cherry is picked randomly from the remaining eighteen. What is the probability that this picked cherry contains a stone?
Hint: You might want to apply the total probability theorem.

(c) (*5 pts*) If instead a monkey gets the original bowl with 20 cherries, and he arranges the cherries in a line and picks the third one and eats it. What is the probability that this picked cherry contains a stone?

(d) (*5 pts*) Given that this third cherry the monkey eats contains a stone, what is the probability that the first two in that line contains at least one stone?

Solution:

(a) Let X denote the number of cherries with stones the pig eats. There are $\binom{20}{2}$ outcomes in sample space and $\binom{5}{k}\binom{15}{2-k}$ outcomes for the event that the two cherries pig has eaten containing k stones. $P(X = k) = \frac{\binom{5}{k}\binom{15}{2-k}}{\binom{20}{2}}$

$$P(X = 0) = \frac{\binom{5}{0}\binom{15}{2-0}}{\binom{20}{2}} = \frac{21}{38} \approx 0.553$$

$$P(X = 1) = \frac{\binom{5}{1}\binom{15}{2-1}}{\binom{20}{2}} = \frac{15}{38} \approx 0.395$$

$$P(X = 2) = \frac{\binom{5}{2}\binom{15}{2-2}}{\binom{20}{2}} = \frac{1}{19} \approx 0.053$$

(b) $P(\text{the 3rd cherry has a stone} \mid X = i) = \frac{5-i}{18}$

According to the total probability theorem,

$$\begin{aligned} P(\text{the 3rd cherry has a stone}) &= \sum_{i=0}^{2} P(\text{the 3rd cherry has a stone} \mid X = i)P(X = i) \\ &= \frac{5}{18}\cdot\frac{21}{38} + \frac{4}{18}\cdot\frac{15}{38} + \frac{3}{18}\cdot\frac{1}{19} = \frac{1}{4} \end{aligned}$$

(c) To put a cherry with stone in third position, we have $\binom{5}{1} = 5$ choices. After picking this one, we can permute all the rest 19 cherries. There are $5 \times 19!$ outcomes in this event and there are $20!$ outcomes in the sample space.

$$P = \frac{5 \times 19!}{20!} = \frac{1}{4}$$

This result is exactly the same as question (b) since their core problems are actually equivalent, which is simply to pick one cherry randomly and see whether it has a stone.

The event that the pig eats two cherries randomly is same as the monkey skips the first two random cherries. In other words, the first step (eating two cherries or skipping the first two) doesn't convey any information or put any constraints on the second step, so whether there is the first step doesn't really matter.

Further, picking the third one in the randomly constructed line is equivalent to randomly picking a cherry in any position.

(d)

$$P(X > 0|B) = \frac{P(X > 0, B)}{P(B)} = 1 - \frac{P(X = 0, B)}{P(B)} = 1 - \frac{\frac{5 \times 15 \times 14 \times 17!}{20!}}{1/4} = \frac{22}{57}$$

Problem 4 *(25pts)*
In the recently concluded world series, suppose that the Los Angeles Dodgers and the Boston Red Sox were evenly balanced, i.e., the probability of each team winning a game was 50% and that every game was **independent** of the others.

(a) *(10 pts)* Given such an analysis, what is the probability that the championship would be **decided in four games**?

(b) *(10 pts)* Given such an analysis, what is the probability that the championship would be **decided in five games**?

(c) *(5 pts)* Suppose that they are tied 2-2 and a highly paid analyst comments that the team that wins the 5th game goes on to win the world series 95% of the time. Your job is to determine whether this analyst really has some expertise. Given the assumption of teams being evenly balanced, what is the probability that the winner of the 5th game wins the world series?

Note: The first team to win four games wins the series.
Solution:

(a) Only if one team wins four games continuously the championship will be decided in four games. Let X_i denote the number of games Los Angeles Dodgers wins in the first i games. $P(X_4 = 4) + P(X_4 = 0) = 2 \times (\frac{1}{2})^4 = \frac{1}{8}$

(b) To satisfy this requirement, in the first five games, one team must win four games and last wining happened at the fifth game. To pick the first three winning among the first four games, we have $\binom{4}{3}$ choices.

$$P(X_4 = 3, X_5 = 4) + P(X_4 = 1, X_5 = 1) = 2 \times \binom{4}{3} \times \left(\left(\frac{1}{2}\right)^3 \times \frac{1}{2}\right) \times \frac{1}{2} = \frac{1}{4}$$

Since there is the ambiguity that "in five games" can be interpreted as "at the fourth game or the fifth game", the answer $\frac{1}{8} + \frac{1}{4} = \frac{3}{8}$ is also acceptable.

Note that $2 \times \binom{5}{4} \times (\frac{1}{2})^5 = \frac{10}{32} = \frac{5}{16}$ is incorrect, because the case of "WWWWL" doesn't exist.

(c) Until fifth game, the winner of the 5th game will win 3 games. To be the champion, he will win in 6th game or win 7th game (the game will end in 7th game anyway).

$$P(\text{the winner of the 5 th game wins the world series}) = \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$$

So the analyst is not reliable.

Problem 5 *(20 pts)*
A player bets on one of the numbers between 1 and 6. Three dice are then rolled, and the number bet by the player appears i times, $i = 0, 1, 2, 3$. The player wins i dollars, if $i = 2, 3$; loses one dollar if $i = 1$, and loses two dollars if $i = 0$. For example, here are two scenarios:

(1) The player bets on the number 5, and the three dice show two 5s, in which case she wins 2 dollars.

(2) The player again bets on 5 and none of the three dice shows 5, in which case she loses 2 dollars.

(a) *(15 pts)* For any particular bet that the player makes (e.g., she bets on 5), determine the probabilities that she

(i) loses 2 dollars, (ii) loses 1 dollar, (iii) wins 2 dollars, and (iv) wins 3 dollars.
Hint: You might want to use binomial law.

(b) *(5pts)* On the average, if the player plays the game a large number of times, will she always win or lose or break even?

Solution:

This is a problem about binomial distribution. After the player's choice of a number, for each die, the probability that the number is rolled out is $\frac{1}{6}$ (no matter which the number is). Thus it is simply a 3 times trial with the probability of "success" being $\frac{1}{6}$. Denote the number of "successes" the player gets in total as X, then $X \sim \textbf{Binomial}(3, \frac{1}{6})$

(a) (i) $P(\text{loses 2 dollars}) = P(X = 0) = \binom{3}{0}(1 - \frac{1}{6})^3 = \frac{125}{216} \approx 0.578704$

(ii) $P(\text{loses 1 dollars}) = P(X = 1) = \binom{3}{1}(\frac{1}{6})(1 - \frac{1}{6})^2 = \frac{75}{216} \approx 0.347222$

(iii) $P(\text{wins 2 dollars}) = P(X = 2) = \binom{3}{2}(\frac{1}{6})^2(1 - \frac{1}{6}) = \frac{15}{216} \approx 0.0694444$

(iv) $P(\text{wins 3 dollars}) = P(X = 3) = \binom{3}{3}(\frac{1}{6})^3 = \frac{1}{216} \approx 0.00462963$

(b) We want to look at the expected value of the winning money to know will the player win or lose money in the long run.

$$
\begin{aligned}
E(\text{the winning money} &= (-2)P(X = 0) + (-1)P(X = 1) + 2P(X = 2) + 3P(X = 3) \\
&= \frac{-2 \times 125 - 75 + 2 \times 15 + 3}{216} \\
&= -\frac{292}{216} \approx -1.35185 < 0
\end{aligned}
$$

So the player is expected to lose money on average, and will break if she plays long enough.

Problem 6 (*35 pts*)
Two weathermen predict weather **independently**, following the probability distribution below:

Truth \ Prediction	Sunny	Rainy
Sunny	0.7	0.3
Rainy	0.2	0.8

(a) The probability distributions of Nicolas's prediction

Truth \ Prediction	Sunny	Rainy
Sunny	0.8	0.2
Rainy	0.1	0.9

(b) The probability distributions of David's prediction

Table 1: The probability distributions of weathermen's prediction

The tables shows the probability of the predictions from each weatherman conditioned on the weather.

For example, the probability $P(\text{Nicolas predicts sunny} \mid \text{It's really sunny}) = 0.7$.

Assume that for each day, the probability of being sunny is 0.9, and the probability of being rainy is 0.1, and the weather of different days are **independent**.

(a) (*5 pts*) If today is a sunny day, then what is the probability that the two weathermen give out the same prediction result?

(b) (*5 pts*) What is the probability that the two weathermen give out the same prediction result for any day (irrespective of the weather)?

(c) (*10 pts*) What is the probability that the first disagreement between the two weathermen occurs at the k^{th} day?
Hint: You might want to use geometric law.

(d) (*5 pts*) What is the probability that they predict correctly for one day given that they agree with each other?

(e) (*10 pts*) Assume they agree with each other for l consecutive days, what is the probability that they predicted correctly m times in these l days?

Solution:

Denote the event that the two weathermen agree with each other and make the same prediction as A, the prediction from Nicolas as N, and the prediction from David as D. Then $N, D \in \{\text{sunny}, \text{rainy}\}$.

(a)

$$\begin{aligned} P(A \mid \text{sunny}) &= P(N = \text{sunny}, D = \text{sunny} \mid \text{sunny}) + P(N = \text{rainy}, D = \text{rainy} \mid \text{sunny}) \\ &= 0.7 \times 0.8 + 0.3 \times 0.2 = 0.62 \end{aligned}$$

(b)

$$\begin{aligned} P(A \mid \text{rainy}) &= P(N = \text{sunny}, D = \text{sunny} \mid \text{rainy}) + P(N = \text{rainy}, D = \text{rainy} \mid \text{rainy}) \\ &= 0.2 \times 0.1 + 0.8 \times 0.9 = 0.74 \end{aligned}$$

Using the total probability theorem,

$$P(A) = P(A \mid \text{sunny})P(\text{sunny}) + P(A \mid \text{rainy})P(\text{rainy}) = 0.62 \times 0.9 + 0.74 \times 0.1 = 0.632$$

Note: one typical error is to calculate $P(N = \text{sunny})$ and $P(D = \text{sunny})$, and then multiply them together to calculate $P(N = \text{sunny}, D = \text{sunny})$. This is incorrect because the event "$N = \text{sunny}$" and "$D = \text{sunny}$" are not independent.

(c) Using geometric law, the probability is

$\big(P(A)\big)^{(k-1)} P(A^{\mathsf{c}}) = 0.632^{(k-1)} \times (1 - 0.632)$

(d)

$$\begin{aligned} & P(N \text{ is correct}, D \text{ is correct} \mid A) \\ =& \frac{P(N \text{ is correct}, D \text{ is correct}, A)}{P(A)} \\ =& \frac{P(N \text{ is correct}, D \text{ is correct})}{P(A)} \\ =& \frac{P(N \text{ is correct}, D \text{ is correct} \mid \text{sunny})P(\text{sunny}) + P(N \text{ is correct}, D \text{ is correct} \mid \text{rainy})P(\text{rainy})}{P(A)} \\ =& \frac{0.7 \times 0.8 \times 0.9 + 0.8 \times 0.9 \times 0.1}{0.632} \approx 0.911392 \end{aligned}$$

(e) This is a binomial distribution. Among l days in which the two weathermen agree with each other, the probability that they predict correctly is 0.911392 for each day, so the answer is

$$\binom{l}{m} 0.911392^m (1 - 0.911392)^{l-m}$$

The credit is given as long as the form is correct, even if the value got from (d) is incorrect.

Problem 7 *(20 pts)*
Extra Credit Problem: Minimal Partial Credit

Shuffle a deck of 52 cards randomly and lay them down one after the other. What is the probability that the first King is immediately followed by the first Ace?

(i) For example, in one instance the first 9 cards are neither an Ace nor a King; the tenth card is the King of Spades and the eleventh card is the Ace of Hearts. Then this will satisfy the condition we ask for.

(ii) In another situation first five cards are neither a King nor an Ace, but the sixth card is the Ace of Clubs. Then this instance does not satisfy the condition we seek.

Solution:
Method 1: Assume the first King occurs in the kth position ($k = 1 \ldots 45$), then there should be no King or Ace before kth position, so we have $\binom{52-8}{k-1}$ choices to arrange them. Then for the first King and Ace, we have $4 \times 4 = 16$ choices. Then we can permuate the rest $52 - (k+1)$ cards. Let A denote the event that the first King is immediately followed by the first Ace.

$$P(A) = \sum_{k=1}^{45} \frac{\binom{44}{k-1} \times (k-1)! \times 16 \times (52 - (k+1)!)}{52!}$$

Method 2: We can calculate the possibility in three steps:

1. Choose a King and an Ace to be that special pair.

   There are in total $4 \times 4 = 16$ choices.

2. For a certain pair of King and Ace, e.g. ♣K and ♡A, the probability that they appear together is calculated as follows:

   We stick the pair together, and treat this pair as a single special card. This makes the number of cards to be $52 - 1 = 51$. The permutation of these 51 cards will exhaust the permutations where ♣K and ♡A are together in this order.

   Thus the probability that a certain picked pair of King and Ace appear together in that order is $\frac{51!}{52!} = \frac{1}{52}$.

3. Given that the special pair of King and Ace appear together, the probability that they appear before the other Kings and Aces is calculated as follows:

   Again we deem this pair of King and Ace together as a single special card. There are 3 other Kings and 3 other Aces, and we want to know **the probability that the special card appears first among these $1 + 3 + 3 = 7$ cards**. Recall that the sample space here is the permutations of 51 cards.

   Since it is equally likely to have each of the 7 cards be the first one among them, the probability is simply $\frac{1}{7}$.

To conclude,

$$P(A) = 16 \times \frac{1}{52} \times \frac{1}{7} = \frac{4}{91}$$