EE 131A — Midterm Solutions
Instructor: Professor Roychowdhury — November 2017

1. Two coins are tossed and the results of this random experiment are reported by a black box. The probabilities of the four different possible outcomes (Coin 1's outcome is the first entry, and Coin 2's outcome is the second entry) are found to be:

$$P(HH) = 1/3, P(TH) = 1/6, P(HT) = 1/6, P(TT) = 1/3.$$

(a)

(i) What is the probability that Coin 1 turns up H?

(ii) What is the probability Coin 2 turns up H?

(iii) What is the probability that the number of Hs = 1 from the two coin tosses?

(b)

(i) What is the conditional probability that Coin 2 is Heads given that Coin 1 is Heads?

(ii) What is the conditional probability that Coin 2 is Heads given that Coin 1 is Tails?

(c) Are the two coin tosses independent? Why or why not?

Solution:

(a) Let $\{C1 = H\}$ be the event that the first coin turns up heads. Since the events $\{HH\}$ and $\{HT\}$ are disjoint we can write

$$P(C1 = H) = P(HH) + P(HT) = \frac{1}{2}$$

In the same fashion we get $P(C2 = H) = P(HH) + P(TH) = \frac{1}{2}$.

Let $\{N_H = 1\}$ be the event that we get exactly one heads. Then

$$P(N_H = 1) = P(TH) + P(HT) = \frac{1}{3}$$

(b)

$$P(C2 = H|C1 = H) = \frac{P(C2 = H \cap C1 = H)}{P(C1 = H)} = \frac{P(HH)}{P(C1 = H)} = \frac{2}{3}$$

By the same token, we get $P(C2 = H|C1 = T) = \frac{P(TH)}{P(C1=T)} = \frac{1}{3}$

(c) As observed in the previous part, $P(C2 = H|C1 = H) \neq P(C2 = H|C1 = T)$, which implies that the outcome of the first coin is not independent of the outcome of the second coin. In contrast, if the coins were independent, then $P(C2 = H|C1 = H) = P(C2 = H|C1 = T) = P(C2 = H)$

2. A primitive village follows a strange custom. They have babies only to have a boy. Whenever a boy is born in a family, they stop having babies and whenever a girl is born, they have a baby again and again till a boy is born. For each child, the probability of having a boy is same as the probability of having a girl.

   (a) What is the probability of having a family of k children?

   (b) What is the expected number of children in a family?

   (c) What is the expected number of girls in a family?

   (d) What fraction of the families have no girls? What is the difference between the expected number of girls in a family and the number of boys in a family? How do you reconcile the two answers?

   Solution:

   (a) Let X be the number of children in a family. Then, the probability that the k^{th} child is a boy(implying that they have k children) is

$$P(X = k) = p(1-p)^{k-1},$$

   where $p = \frac{1}{2}$.

   (b) We know that the expected value of a geometric random variable is $\frac{1}{p}$, which in this case implies that $E[X] = 2$.

   (c) Since X is the number of children in a family, the $Y = X - 1$ is the number of girls in the family and we have

$$E[Y] = E[X] - 1 = 1.$$

   Or, you can do it the hard way and compute

$$E[Y] = \sum_{k=1}^{\infty}(k-1)p(X=k) = \sum_{k=1}^{\infty} k(\frac{1}{2})^k - \sum_{k=1}^{\infty} p(X=k) = 2 - 1 = 1$$

   (d) Every family will eventually have a boy. In the expected sense, half of the families will only have a boy and no girl. Despite this, the expected number of girls in a family is the same as the number of boys although half of the families do not have any girls. This can be understood by noting that the larger the family the more girls they have.

3. In London, half of the days have some rain. The weather forecaster is correct 2/3 of the time, i.e., the probability that it rains, given that she has predicted rain, and the probability that it does not rain, given that she has predicted that it won't rain, are both equal to 2/3. A man takes his umbrella always, when rain is forecast. When rain is not forecast, he takes it with probability 1/3.

   (a) Find the probability that the man doesn't bring his umbrella.

   (b) Find the probability that the man doesn't have his umbrella, given that it rains.

(c) Find the probability that it doesn't rain, given that he brings his umbrella.

Solution: Let R be the event that it rains in a given day and let F be the event that the rain was predicted and U be the event that the man takes his umbrella. Let us follow the assumption that $P(R|F) = P(R^c|F^c) = \frac{2}{3}$

$$P(R) = P(R|F)P(F) + P(R|F^c)P(F^c) = P(R|F)P(F) + (1 - P(R^c|F^c))(1 - P(F)),$$

which gives

$$\frac{1}{2} = \frac{2}{3}P(F) + \frac{1}{3}(1 - P(F)),$$

which yields that $P(F) = \frac{1}{2}$. By the Bayes law, it is easy to see that we could alternatively assume that $P(F|R) = P(F^c|R^c) = \frac{2}{3}$.

(a) By the law of total probability

$$P(U^c) = P(U^c|F)P(F) + P(U^c|F^c)P(F^c) = (1 - P(U|F^c)) \times \frac{1}{2} = \frac{1}{3}$$

(b) By the total probability theorem in the space conditioned on the Rain event

$$P(U^c|R) = P(U^c|FR)P(F|R) + P(U^c|F^cR)P(F^c|R) = P(U^c|F^c)P(F^c|R) = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$$

(c) Using the Bayes law

$$P(R^c|U) = \frac{P(U|R^c)P(R^c)}{P(U)} = \frac{\frac{5}{9} \times \frac{1}{2}}{\frac{2}{3}} = \frac{5}{12},$$

where $P(U|R^c)$ is obtained from a similar computation as in part (b) as follows.

$$\begin{aligned} P(U|R^c) &= P(U|FR^c)P(F|R^c) + P(U|F^cR^c)P(F^c|R^c) \\ &= P(U|F)P(F|R^c) + P(U|F^c)P(F^c|R^c) \\ &= 1 \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} = \frac{5}{9} \end{aligned}$$

4. A standard deck of cards includes 13 denominations ordered in the following ranks: 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, and Ace, and in 4 suits: Hearts, Spades, Diamonds, Clubs. In the game of poker, a hand is a set of 5 cards (no order) given to a player. A Royal Flush is a hand that has Ace, King, Queen, Jack, and 10 all from the same suit (e.g. all of Hearts). A Two Pairs is a hand with two pairs of cards of the same denomination, and another card of a different denomination (e.g. 3 of Clubs, 3 of Spades, 10 of Clubs, 10 of Hearts, King of Diamonds).

   (a) Assume that a player is given the five cards of a hand sequentially, i.e. he/she is given 5 cards, one at a time. Find the probability that the fourth card is the first King dealt, meaning that the 3 previous cards are non-Kings and the fourth one is a King).

(b) Find the probability that a player is given a Two Pairs hand.

(c) Assume that from the same full deck, we deal a hand to two players. Find the probability of a Royal Flush hand for the first player. Given that the first player is given a Royal Flush, find the conditional probability that the second player is given a Royal Flush.

Solution:

(a) The first three cards should be non-Kings and the fourth one be a King. The fifth one is not important. The answer therefore is:

$$\frac{48}{52} \times \frac{47}{51} \times \frac{46}{50} \times \frac{4}{49} \times 1$$

(b)

$$\frac{\binom{13}{2} \times \binom{11}{1} \times \binom{4}{2} \times \binom{4}{2} \times \binom{4}{1}}{\binom{52}{5}}$$

(c) Considering the reduced sample space, regardless of the suite of the first player's hand, we have:

$$\frac{\text{number of remaining royal flushes}}{\text{number of remaining hands}} = \frac{3}{\binom{52-5}{5}}$$

5. The number of incoming calls received per minute at a hotels reservation center is a Poisson random variable with mean 3.

(a) Find the probability that no calls arrive in a given 1-minute period.

(b) Find the probability that at least two calls will arrive in a given 2-minute period.

(c) Suppose that in a given minute 3 calls were received. What is the conditional probability that all 3 calls were made in the first 30 seconds?

Solution:

(a)

$$X \sim \mathit{Poisson}(3)$$

$$P(X = 0) = \frac{e^{-3}3^0}{0!} = e^{-3}$$

(b)

$$X \sim \mathit{Poisson}(6)$$

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-6} - \frac{e^{-6}6^1}{1!} = 1 - 7e^{-6}$$

(c) Let B be the event that 3 calls arrive in a given one minute time interval, and let A be the event the these calls all arrive in the first 30 seconds; meaning that there are three in the first 30 seconds and none in the second 30 seconds. Thus:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

In the first and second 30 seconds we have two independent Poisson random variables X and Y with averages 1.5 calls. The desired probability is therefore:

$$P(A) = P(X = 3) \times P(Y = 0) = \frac{e^{-1.5} 1.5^3}{3!} \times e^{-1.5} = \frac{1}{8} \times \frac{e^{-3} 3^3}{3!}$$

$$P(B) = \frac{e^{-3} 3^3}{3!}$$

$$P(A|B) = \frac{1}{8}$$

We can justify this answer as follows: each of the 3 calls were equally likely to have happened either in the first 30 seconds or in the second 30 seconds; so the conditional probability that each call occurs on the first 30 minutes is $\frac{1}{2}$. Note also that each call arrives independently from the others. So the chance that all of them happened in the first half minute is $(\frac{1}{2})^3 = \frac{1}{8}$.

6. If someone picks two cards sequentially from a full deck of cards (refer to problem 4 for the definition of a standard deck of cards), what is the probability that the rank of the second card is higher than that of the first one? (e.g. the first one is the 4 of Clubs and the second one is the 10 of Spades)

Solution:

$$\begin{aligned}
&P(\text{rank of the second card } > \text{ rank of the first card}) \\
&= \sum_{i=1}^{13} P(\text{rank of first card} = i) P(\text{rank of second card} > i \mid \text{rank of first card} = i) \\
&= \sum_{i=1}^{13} \frac{1}{13} \frac{4(13-i)}{51} \\
&= \frac{4}{13 \times 51} \sum_{i=1}^{13} (13 - i) \\
&= \frac{4}{13 \times 51} (13^2 - \sum_{i=1}^{13} i) \\
&= \frac{8}{17}
\end{aligned}$$

Another solution to this problem is through symmetry. By $\frac{3}{51}$ chance, the second card is of the same rank as the first card. If not however, it is equally likely that the second card is either of a higher or a lower rank than the first card. Therefore:

$$P(\text{rank of the second card } > \text{ rank of the first card}) = \frac{1 - \frac{3}{51}}{2} = \frac{8}{17}$$