1. In a game of blackjack two cards are picked at random from a deck of 52 cards. Note that a deck of cards has 13 denominations (or kinds) (namely, they are ordered as Ace, $2, 3, 4, 5, 6, 7, 8, 9, 10,$ Jack, Queen, and King) and 4 suits (namely, Hearts, Spades, Clubs, and Diamonds).

   (a) What is the probability of being dealt a blackjack? That is, when one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king. So, 10 of Spades, and Ace of Clubs will comprise a blackjack.

   (b) Suppose that you are playing blackjack against a dealer. Each one of you are dealt 2 cards from a deck of 52 cards (without replacing any cards). What is the probability that both you and the dealer are dealt a blackjack?

   (c) What is the probability that neither you nor the dealer is dealt a blackjack?

   Solution:

   (a) Drawing without replacement

$$P(\text{blackjack}) = \frac{\binom{4}{1}\binom{16}{1}}{\binom{52}{2}}$$

   (b) Define $A =$ You are dealt a blackjack and $B =$ The dealer is dealt a blackjack. Then

$$P(A \cap B) = 4 \times \frac{4 \times 16}{52 \times 51} \times \frac{3 \times 15}{50 \times 49}$$

   (c) Using De' Morgan's law, one gets $P(A^c \cap B^c) = 1 - P(A \cup B)$. Further note that, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. In this setting, due to symmetry, $P(A) = P(B)$. $P(A)$ and $P(A \cap B)$ are evaluated in parts (a) and (b). Thus,

$$P(A^c \cap B^c) = 1 - 2(\frac{\binom{4}{1}\binom{16}{1}}{\binom{52}{2}}) + 4 \times \frac{4 \times 16}{52 \times 51} \times \frac{3 \times 15}{50 \times 49} = 0.9017$$

2. A person tried by a 3-judge panel is declared guilty if at least 2 judges cast votes of guilty. Suppose that when the defendant is in fact guilty, each judge will independently vote guilty with probability 0.7, whereas when the defendant is in fact innocent, this probability is 0.2. Let 75 percent of defendants be guilty.

   (a) What is the probability of a guily verdict?

   (b) Given that the verdict is "guilty", what is the conditional probability that the defendant was indeed guilty? Is the system fair?

   (c) Compute the probability that judge number 3 votes "guilty" given that judges 1 and 2 vote "guilty".

   Solution:

(a) Let A be the event the defendant is guilty and let B be the event that the verdict is guilty. Let X be the number of that vote guilty. Then, $B = \{X \geq 2\}$. Using law of total probabilities

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$

We compute $P(B|A)$:

$$P(B|A) = P(X \geq 2|A) = p(X = 2|A) + P(X = 3|A) = \binom{3}{2}(0.7)^2(0.3) + (0.7)^3.$$

Likewise,

$$P(B|A^c) = \binom{3}{2}(0.2)^2(0.8) + (0.2)^3.$$

So, $P(B) = 0.75 \times 0.784 + 0.25 \times 0.104 = 0.614$.

(b) As always, we use Bayes' law for such problems

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{0.75 \times 0.784}{0.614} = 0.958$$

When the system declares a defendant as guilty, with high probability the defendant is indeed guilty, so the system is quite fair.

(c) Let B_i be the event that judge i votes guilty. Then

$$P(B_3|B_1, B_2) = \frac{P(B_3 \cap B_1 \cap B_2)}{P(B_1 \cap B_2)} = \frac{P(X = 3)}{P(B_1 \cap B_2)}.$$

To evaluate these probabilities, we condition on the defendant being guilty. We can then use what we computed in part (a):

$$P(X = 3) = P(X = 3|A)P(A) + P(X = 3|A^c)P(A^c) = 0.75 \times (0.7)^3 + 0.25(0.2)^3$$

and

$$\begin{aligned} P(B_1 \cap B_2) &= P(B_1 \cap B_2|A)P(A) + P(B_1 \cap B_2|A^c)P(A^c) \\ &= 0.75 \times (0.7)^2 + 0.25 \times (0.2)^2. \end{aligned}$$

Thus,

$$P(B_3|B_1, B_2) = \frac{P(X = 3)}{P(B_1 \cap B_2)} = 0.731$$

3. At a given time, the number of users trying to get access to a shared frequency band is a Poisson random variable with mean 10. Suppose that the total available bandwidth is 100 MHz and is equally divided among the users; for example when there are 5 users, each will get 20 MHz of bandwidth.

   (a) Let X be the bandwidth per user in this network. Find the probability distribution of X. Note that the sample space for X is $\{100\text{MHz}, 50\text{MHz}, 100/3\text{MHz}, \cdots\}$.

(b) Find the minimum bandwidth that is allocated to a transmitting user with probability 90 percent or higher.

(c) What is the probability that a user has a share of 10 MHZ of bandwidth or higher?

Solution:

(a) Let N be the number of users requesting access to the frequency band. Then one can see that $X = 100/N$ Therefore,

$$P(X = 100/k) = P(N = k) = \frac{10^k e^{-10}}{k!} \qquad k \geq 1$$

(b) Finding the maximum bandwidth that can be guaranteed with probability more than 0.9 is equivalent to $P(X \geq \frac{100}{k_{max}}) \geq 0.9$. But,

$$P(X \geq \frac{100}{k_{max}}) = P(N \leq k_{max}) = \sum_{i=0}^{k_{max}} P(N = i) \geq 0.9$$

Then, it turns out that $k_{max} = 14$.

(c)

$$P(X \geq 10) = P(N \leq 10) = \sum_{i=0}^{10} P(N = i) \sim 0.58$$

4. There are $k+1$ distinct channels in a communication system. The probability of error on the i^{th} channel is i/k, for $i = 0, 1, \cdots, k$. Thus the 0th channel commits no errors; the first channel commits errors with probability $1/k$ etc. Suppose a transmitter first selects one of the channels randomly and keeps transmitting messages on the same channel. If all the first n transmissions are received with error:

(a) What is the conditional probability that the i^{th} channel was used?

(b) what is the conditional probability that $(n+1)^{st}$ transmission is received with error?

Solution:

(a) Let A_i be the event that i^{th} channel was used and let B be the event that the first n transmissions are in error. We want to compute $P(A_i|B)$ and we use Bayes' rule to do so:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)}$$

Since all channels are equally likely to be selected $P(A_i) = \frac{1}{k+1}$. Moreover, given that channel i was selected the probability that all first n transmissions were on error is $P(B|A_i) = (\frac{i}{k})^n$ Then, using the law of total probabilities

$$P(B) = \sum_{i=0}^{k} P(B|A_i)P(A_i) = \sum_{i=0}^{k} (\frac{i}{k})^n \frac{1}{k+1}.$$

Thus

$$P(A_i|B) = \frac{(\frac{i}{k})^n \frac{1}{k+1}}{\sum_{j=0}^{k} (\frac{j}{k})^n \frac{1}{k+1}} = \frac{i^n}{\sum_{j=1}^{k} j^n}$$

(b) Let C be the event that the $(n+1)^{th}$ transmission is on error. Then

$$P(C|B) = \sum_{i=0}^{k} P(C|B, A_i) P(A_i|B).$$

Note that $P(C|B, A_i) = P(C|A_i) = \frac{i}{k}$ and $P(A_i|B)$ is computed in part (b). Thus,

$$P(C|B) = \frac{\sum_{i=0}^{k} i^{n+1}}{k \sum_{j=0}^{k} j^n}$$

5. A Hacker is trying to break into a system. She has a list of n possible passwords to check, and exacly one of them is the correct password.

(a) If she tries the passwords at random, each time discarding those that do not work (i.e., without replacement), what is the probability that she will access the system on her k^{th} try?

(b) If she does not discard the previously tried passwords (that is, each time she tries a randomly picked password from the entire list of n passwords) then what is the probability that she will access the system on her k^{th} try?

(c) Note that in the setting of part (a), she will certainly find the right password in at most n trials. Compared to that, in part (b),

(i) What is the expected number of attempts before she breaks into the system?

(ii) What is the probability that she will need more than n trials? What value does this probability of extra attempts converges to when n is very large?

Solution:

(a) Let A_i be the event that the i^{th} password works. Since the false passwords are discarded, the probability of success in the k^{th} try is

$$\begin{aligned} P(A_i \cap A_{i-1}^c \cap \cdots A_1^c) &= P(A_1^c)P(A_2^c|A_1^c)P(A_3^c|A_2^c, A_1^c) \cdots P(A_i|A_{i-1}^c, \cdots, A_i^c) \\ &= \frac{n-1}{n} \times \frac{n-2}{n-1} \times \cdots \times \frac{1}{n-k+1} = \frac{1}{n} \end{aligned}$$

(b) Since there is replacement:

$$\begin{aligned} P(A_i \cap A_{i-1}^c \cap \cdots A_1^c) &= P(A_1^c)P(A_2^c|A_1^c)P(A_3^c|A_2^c, A_1^c) \cdots P(A_i|A_{i-1}^c, \cdots, A_i^c) \\ &= \frac{n-1}{n} \times \frac{n-1}{n} \times \cdots \times \frac{1}{n} = (1 - \frac{1}{n})^{k-1} \frac{1}{n} \end{aligned}$$

(c) (i) Let X be the number of attempts. As shown in part (b), X is a geometric random variable, thus $E[X] = n$

(ii)There are two ways to do this problem. Note that requiring n trials or more is simply equivalent to having the first $n-1$ rounds all failed. Thus,

$$P(X \geq n) = P(\text{failure in the first } n-1 \text{ rounds}) = (1 - \frac{1}{n})^{n-1}.$$

You can do it in a tedious way and compute

$$P(X \geq n) = \sum_{i=n}^{\infty} P(X = i) = \frac{1}{n} \sum_{i=n}^{\infty} (1 - \frac{1}{n})^k = (1 - \frac{1}{n})^{n-1}$$

Letting n to grow to infinity, yields

$$\lim_{n \to \infty} P(X \geq n) = e^{-1}$$

6. A deck of 52 playing cards, as described in problem 1, is shuffled and the cards turned up one at a time. What is the probability that the first king shows up right after the first ace?

   Solution: Note that turning up cards one by one in order corresponds to a permutation of cards and there are in total 52! such permutations. Among all these permutations we are interested in those in which the first ace is followed by the first King. Suppose that the first ace appears in the k^{th} position in the permutation followed by the first king in the position $k+1$. The choice of the first ace and first king can be done in 4×4 possible ways. Among all 44 non-King, non-Ace cards we need to put $k-1$ of them in the first $k-1$ positions. This can be done in $\binom{44}{k-1}(k-1)!$ ways. Now it remains $52-(k+1)$ cards including 3 aces and 3 Kings and the remaining $44-(k-1)$ non-King, non-Ace cards, to be arranged after position $k+1$, which can be done in $(51-k)!$ ways. Now we only need to sum up over all possibilities for k, the position of the first ace, which can range from $1, \cdots, 45$.

$$P(\text{first ace followed by the first king}) = \sum_{i=1}^{45} \frac{16 \times \binom{44}{k-1}(k-1)! \times (51-k)!}{52!}$$

An equivalent solution would be to take out all the Aces and Kings, choose the first Ace and King, bind them together and arrange these 7 objects in 51 slots such that the the the (first Ace, first King) pair comes before the other Kings and Aces. There are $\binom{51}{7}$ ways to determine the position of Kings and Aces. putting the pair in the first position there are 6! ways to permute the other Aces and Kings in the positions chosen. Now there are 44! ways to arrange the rest of the cards in the remaining positions. Thus

$$P(\text{first ace followed by the first king}) = \frac{16\binom{51}{7}6!44!}{52!}$$

Note that the solutions above are equivalent and in fact it is an interesting exercise to show that two quantities are the same.