EE 131A | Midterm Solutions
Probability | Fall 2014
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1. In a game of poker five cards are picked at random from a deck of 52 cards. Note that a deck of cards has 13 denominations (or kinds) (namely, they are ordered as Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, and King) and 4 suits (namely, Hearts, Spades, Clubs, and Diamonds). What is the probability of being dealt

   (a) A full house? That is, when the cards have denominations a, a, a, b, b, where a and b are distinct. So, 10 of Spades, 10 of Hearts, 10 of Diamonds, Queen of Hearts and Queen of Clubs will comprise a full house.

   (a) A Straight? This occurs when the cards have distinct conescutive denominations but not all of the same suit. So Ace of Spades, 2 of Hearts, 3 of Spades, 4 of Spades, and 5 of Spades will comprise a Straight. Note that Ace can be regarded as both the least or the greatest value, so 10 of Clubs, Jack of Clubs, Queen of Diamonds, King of Diamonds, and Ace of Diamonds will also comprise a Straight.

   Solutions:

   (a) There are $\binom{52}{5}$ total number of hands you can get with equal probability. To calculate how many of them are full house hands, we need to first choose a and b from 13 possible denominations and then choose a suit for each card:

$$P(full\ house) = \frac{13 \times 12 \times \binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

   (b) To calculate the number of straights, we choose the starting point of the straight which can be one of Ace, 2, 3, ..., 10, and then find the possible combination of suits except the ones where all the suits are the same($4^5 - 4$):

$$P(straight) : \frac{10 \times (4^5 - 4)}{\binom{52}{5}}$$

2. Suppose you roll two fair dice. If the sum is greater than or equal to 10 you stop, but if the sum is less than 10 you roll the two dice one more time and then stop (so you either roll once or twice). If you stop the game with a sum less than 10, you lose \$10, but if you stop the game with a sum greater than or equal to 10, you win 3 times the amount of the sum in dollars.

   For example, (i) in one scenario in your first roll you might get a sum of 11, then you stop and you win \$33. (ii) in another scenario, in your first roll you might get a sum of 3, then you are forced to roll again, and say you get a sum of 12 in your second roll; then you win \$36, (iii) in yet another scenario, you might get a sum of 5 in your first roll, then roll again, and might get a sum of 4; then you lose \$10.

(a) What is the probability of stopping after the first roll. What about the probability of having to roll twice? Hint: You may first want to calculate the probability of getting a sum of x in any roll of two fair dice, for x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

(b) What is the probability of stopping with a sum of 11 (12)? What is the probability of stopping with a sum of 3?

(c) If a player stops with a sum of 11 (or 12), what is the probability they have rolled twice? (Hint: Use Bayes Law).

(d) What is the probability the player has rolled twice if they stopped with a sum of 3?

(e) What is the expected value of your money winnings in this game?

Solutions:

(a) Let S be the sum in a toss of two dice and N_T the number of tosses:

$$P(N_T = 2) = P(S \geq 10) = \sum_{i=10}^{12} P(S = i) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{6}$$

$$P(N_T = 1) = 1 - P(N_T = 2) = \frac{5}{6}$$

(b) If R is the final result sum:

$$P(R = 11) = P(S = 11) + P(N_T = 2) \times P(S = 11) = \frac{2}{36} + \frac{5}{6} \times \frac{2}{36} = \frac{11}{108}$$

$$P(R = 12) = P(S = 12) + P(N_T = 2) \times P(S = 12) = \frac{1}{36} + \frac{5}{6} \times \frac{1}{36} = \frac{11}{216}$$

$$P(R = 3) = P(N_T = 2) \times P(S = 3) = \frac{5}{6} \times \frac{2}{36} = \frac{5}{108}$$

(c) Using Bayes' Rule:

$$P(N_T = 2|R = 11) = \frac{P(R = 11|N_T = 2)P(N_T = 2)}{P(R = 11)} = \frac{\frac{2}{36} \times \frac{5}{6}}{\frac{11}{108}} = \frac{5}{11}$$

$$P(N_T = 2|R = 12) = \frac{P(R = 12|N_T = 2)P(N_T = 2)}{P(R = 12)} = \frac{\frac{1}{36} \times \frac{5}{6}}{\frac{11}{216}} = \frac{5}{11}$$

(d) $P(N_T = 2|R = 3) = \dfrac{P(R = 3|N_T = 2)P(N_T = 2)}{P(R = 3)} = \dfrac{\frac{2}{36} \times \frac{5}{6}}{\frac{5}{108}} = 1$

Which means if the result sum is 3 we have tossed twice for sure.

(e) Let W be the winnings from the game:

$$E[W] = P(R < 10) \times (-10) + \sum_{i=10}^{12} P(R = i) \times (3i)$$

$$P(R=10) = P(S=10) + P(N_T=2)\times P(S=10) = \frac{3}{36} + \frac{5}{6}\times\frac{3}{36} = \frac{33}{216}$$

Therefore:

$$E[W] = \frac{5}{6}.\frac{5}{6}(-10) + \frac{33}{216}\times 30 + \frac{22}{216}\times 33 + \frac{11}{216}\times 36 = \frac{17}{6}$$

3. A man has 5 coins in his pocket. Two are double-headed, one is double-tailed, and two are normal. The coins cannot be distinguished unless one looks at them.

   (a) The man shuts his eyes, chooses a coin at random, and tosses it. What is the probability that the lower face of the coin is heads?

   (b) He opens his eyes and sees that the upper face of the coin is a head. What is the probability that the lower face is a head.

   (c) He shuts his eyes again, picks up the same coin, and tosses it again. What is the probability that the lower face is a head?

   (d) He opens his eyes and sees that the upper face is a head. What is the probability that the lower face is a head?

   Solutions:
   Let A denote the event that he picks a double-headed coin, B denote the event that he picks a normal coin, and C be the event that he picks the double-tailed coin. Let H_{L_i} (and H_{L_i}) denote the event that the lower face (and the upper face) of the coin on the ith toss is a head.

   (a)

$$\begin{aligned} P(H_{L_1}) &= P(H_{L_1}|A)P(A) + P(H_{L_1}|B)P(B) + P(H_{L_1}|C)P(C) \\ &= 1\times\frac{2}{5} + \frac{1}{2}\times\frac{2}{5} + 0\times\frac{1}{5} = \frac{3}{5} \end{aligned}$$

   (b)

$$P(H_{L_1}|H_{U_1}) = \frac{P(H_{L_1}\cap H_{U_1})}{P(H_{U_1})} = \frac{2/5}{3/5} = \frac{2}{3}$$

   (c)

$$\begin{aligned} P(H_{L_2}|H_{U_1}) &= \frac{P(H_{L_2}\cap H_{U_1})}{P(H_{U_1})} \\ &= \frac{P(H_{L_2}\cap H_{U_1}|A)P(A) + P(H_{L_2}\cap H_{U_1}|B)P(B) + P(H_{L_2}\cap H_{U_1}|C)P(C)}{P(H_{U_1})} \\ &= \frac{1\times\frac{2}{5} + \frac{1}{4}\times\frac{2}{5} + 0\times\frac{1}{5}}{3/5} = \frac{5}{6} \end{aligned}$$

(d)

$$\begin{aligned} P(H_{L_2}|H_{U_1} \cap H_{U_2}) &= \frac{P(H_{L_2} \cap H_{U_1} \cap H_{U_2})}{P(H_{U_1} \cap H_{U_2})} \\ &= \frac{1 \times \frac{2}{5} + 0 \times \frac{2}{5} + 0 \times \frac{1}{5}}{1 \times \frac{2}{5} + \frac{1}{4} \times \frac{2}{5} + 0 \times \frac{1}{5}} = \frac{4}{5} \end{aligned}$$

4. A Christmas fruitcake has Poisson-distributed independent number of sultana raisins, iridescent red cherry bits, and radioactive green cherry bits with respective averages 48, 24, and 12 bits per cake. Suppose you politely accept 1/12 of a slice of the cake.

(a) What is the probability that you get lucky and get no green bits in your slice?

(b) What is the probability that you get really lucky and get no green bits and two or fewer red bits in your slice?

(c) What is the probability that you get extremely lucky and get no green or red bits and more than five raisins in your slice?

Solution:

(a) Let S, R, and G be the number of sultana raisins, red bits, and green bits respectively. Each one of these variable has a poisson distribution with $\alpha = 48, 24, 12$. If we randomly pick 1/12 of the cake the number of bits in the slice will now have poisson distributions with averages of $\alpha = 4, 2, 1$ which are 1/12 of the average number of bits in the entire cake. So S_s, R_s, and G_s which are the number of bits in the slice will have the following probability distributions:

$P(S_s = k) = \frac{4^k}{k!} e^{-4}$

$P(R_s = k) = \frac{2^k}{k!} e^{-2}$

$P(G_s = k) = \frac{1^k}{k!} e^{-1}$

Now the probability that we get no green bits is equal to:

$P(G_s = 0) = e^{-1}$

(b)

$P(G_s = 0 \text{ and } R_s \leq 2) = P(G_s = 0) \sum_{k=0}^{2} P(R_s = k) = e^{-1}(e^{-2} + \frac{2}{1}e^{-2} + \frac{4}{2}e^{-2}) = 5e^{-1}e^{-2} = 5e^{-3}$

(c)

$P(G_s = 0 \text{ and } R_s = 0 \text{ and } S_s > 5) = P(G_s = 0)P(R_s = 0) \left(1 - \sum_{k=0}^{5} P(S_s = k)\right) =$

$$e^{-1}e^{-2}\left(1-(e^{-4}+\tfrac{4}{1!}e^{-4}+\tfrac{4^2}{2!}e^{-4}+\tfrac{4^3}{3!}e^{-4}+\tfrac{4^4}{4!}e^{-4}+\tfrac{4^5}{5!}e^{-4})\right)$$

5. You choose a positive integer less than 1000 uniformly at random. Find the probability that sum of the digits is 8 (or 9).

   Solutions:
   Let x_1, x_2 and x_3 represent the first, second and third digits of the number(leave zero for the missing digits if not a 3-digits number). Then the number of such integers is the number of non-negative solutions to the equation $x_1 + x_2 + x_3 = 8$ which is $\binom{8+2}{2}$ (or, for a sum of 9, $\binom{9+2}{2}$).